Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

2.5: Closer Look at the Euler Method

Euler Method Error:

If $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ has a unique solution on $[a,b]$, and we have the Euler approximations $y_1, y_2, \ldots, y_k$ to the actual values $y(x_1), y(x_2), \ldots, y(x_k)$ which have been computed using step size h, then the error $|y_i - y(x_i)| \leq h \cdot C$ for some $C > 0$.

The value C (and therefore the amount of error) increases as the maximum value of $|y''(x)|$ increases on $[a,b]$.

Improvement in Euler Method: Apply the iterative formulas…

$$k_1 = f(x_n, y_n),$$
$$u_{n+1} = y_n + h \cdot k_1,$$
$$k_2 = f\left(x_{n+1},\ u_{n+1}\right)$$
$$y_{n+1} = y_n + h \cdot \tfrac{1}{2}(k_1 + k_2).$$

To obtain approximations $(y_1, y_2, \ldots)$ of the actual values $(y(x_1), y(x_2), \ldots)$.

Note: While the improved method may reduce the number of approximations $(y_1, y_2, \ldots)$ needed to reach some level of accuracy, since it involves more calculations for every approximation, it is not guaranteed that a computer program can obtain the increased accuracy in less time than the original method would have. However, it *is* known that if the desired solution y has a continuous third derivative, then the improved Euler method is indeed more efficient at obtaining accuracy. In a great many practical applications, this requirement is known to be satisfied, even if the exact solution y is not yet known.

Exercises

Problem: #2

Apply the improved Euler method to approximate the following differential equation on $\left[0,\ 0.5\right]$ with step size $h = 0.1$. Construct a table showing four-decimal-place values of the approximate solution and actual solution at the points $x = 0.1,\ 0.2,\ 0.3,\ 0.4,$ and 0.5.

$$y' = 2y, \qquad y(0) = \tfrac{1}{2}; \qquad y(x) = \tfrac{1}{2}e^{2x}.$$

Apply the iterative formulas…

$$k_1 = f(x_n, y_n),$$
$$u_{n+1} = y_n + h \cdot k_1,$$
$$k_2 = f\left(x_{n+1},\ u_{n+1}\right)$$
$$y_{n+1} = y_n + h \cdot \tfrac{1}{2}(k_1 + k_2).$$

$$y_1 = y_0 + h \cdot \tfrac{1}{2}(k_1 + k_2) = \tfrac{1}{2} + \tfrac{1}{20}(k_1 + k_2)$$

$$k_1 = f(x_0, y_0) = f\left(0, \tfrac{1}{2}\right) = 2 \cdot \tfrac{1}{2} = 1.$$

$$u_1 = y_0 + \tfrac{1}{10} \cdot 1 = \tfrac{1}{2} + \tfrac{1}{10} = \tfrac{6}{10}.$$

$$k_2 = f(x_1, u_1) = f\left(\tfrac{1}{10}, \tfrac{6}{10}\right) = 2 \cdot \tfrac{6}{10} = \tfrac{12}{10}.$$

$$y_1 = \tfrac{1}{2} + \tfrac{1}{20}\left(1 + \tfrac{12}{10}\right) = \tfrac{1}{2} + \tfrac{22}{200} = \tfrac{122}{200} = 0.61.$$

$y_2 = 0.61 + \frac{1}{20}(k_1 + k_2)$

$k_1 = f(x_1, y_1) = f(0.1, 0.61) = 2 \cdot 0.61 = 1.22.$

$u_2 = y_1 + \frac{1}{10} \cdot 1.22 = 0.61 + 0.122 = 0.732.$

$k_2 = f(x_2, u_2) = f(0.2, 0.732) = 2 \cdot 0.732 = 1.464.$

$y_2 = 0.61 + \frac{1}{20}(1.22 + 1.464) = 0.7442$

Etc.…

x	0.0	0.1	0.2	0.3	0.4	0.5
y with $h = 0.1$	0.5000	0.6100	0.7422	0.9079	1.1077	1.3514
y actual	0.5000	0.6107	0.7459	0.9111	1.1128	1.3591

Problem: #**6**

Same as the previous problem except with…

$y' = -2xy, \qquad y(0) = 2; \qquad y(x) = 2e^{-x^2}.$

$y_1 = y_0 + h \cdot \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{20}(k_1 + k_2)$

$k_1 = f(x_0, y_0) = f(0, 2) = -2 \cdot 0 \cdot 2 = 0.$

$u_1 = y_0 + \frac{1}{10} \cdot 0 = 2.$

$k_2 = f(x_1, u_1) = f\left(\frac{1}{10}, 2\right) = -2 \cdot \frac{1}{10} \cdot 2 = -\frac{2}{5}.$

$y_1 = 2 + \frac{1}{20}\left(0 - \frac{2}{5}\right) = 1.98.$

$y_2 = y_1 + \frac{1}{20}(k_1 + k_2)$

$k_1 = f(x_1, y_1) = f(0.1, 1.98) = -2 \cdot 0.1 \cdot 1.98 = -0.396$

$u_2 = y_1 + \frac{1}{10} \cdot -0.396 = 1.98 - 0.0396 = 1.9404$

$k_2 = f(x_2, u_2) = f(0.2, 1.9404) = -2 \cdot 0.2 \cdot 1.9404 = -0.77616$

$y_2 = 1.98 + \frac{1}{20}(-0.396 - 0.77616) = 1.921392$

Etc.…

x	0.0	0.1	0.2	0.3	0.4	0.5
y with $h = 0.1$	2.0000	1.9800	1.9214	1.8276	1.7041	1.5575
y actual	2.0000	1.9801	1.9216	1.8279	1.7043	1.5576