Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Autonomous DEQs
- Critical Pts/Equilibrium Sols
- Stable/Unstable Critical Pts
- Bifurcation Pts/Diagram

2.4: The Euler Method

Unsolvable DEQs: $\frac{dy}{dx} = f(x,y)$, with initial condition $y(x_0) = y_0$.

What if our methods for solving 1st-order DEQs still fail us? (And they often do!)

How do we use our knowledge of slope fields to generate solution curves?

Definition: an **elementary function** is a function that is made up of a *finite* number of arithmetic operations $(+\ -\ \times\ \div)$, constants, exponentials, logarithms, or trigonometric functions. In other words, elementary functions are **most everything you've ever seen in math**!

What if the soln to a DEQ isn't elementary?

Recall that all our methods so far have produced sols consisting of elementary functions.

Example: The anti-derivative of $y' = e^{-x^2}$ is non-elementary, so you can't easily solve for it.

Other non-elementary examples when: $f(x) = |x|$, $f(x) = \int_0^x e^{-t^2}dt$, $f(x) = \int_0^x \ln(\ln x)dt$, $f(x) = \int_0^x \sin(x^2)dt$, etc.

However, we can usually generate an aproximation, and perhaps that's sufficient for your purposes.

But how? We make use of the slope field. Remember, the slope field tells us where the solution is headed!

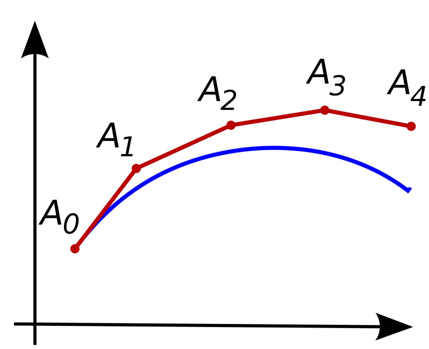
Method: Start at some (x_0,y_0), check direction field there $f(x_0,y_0)$, then move in that direction for some distance h. Giving us:

$y_1 = y_0 + h \cdot f(x_0,y_0)$ and $x_1 = x_0 + h$. (recall: rise = run$\cdot\frac{\text{rise}}{\text{run}}$ or rise = $h \cdot f$)

Next, check direction field again $f(x_1,y_1)$, and move in new direction for h.

$y_2 = y_1 + h \cdot f(x_1,y_1)$ and $x_2 = x_0 + 2h$,

Continue process till you have generated your curve.

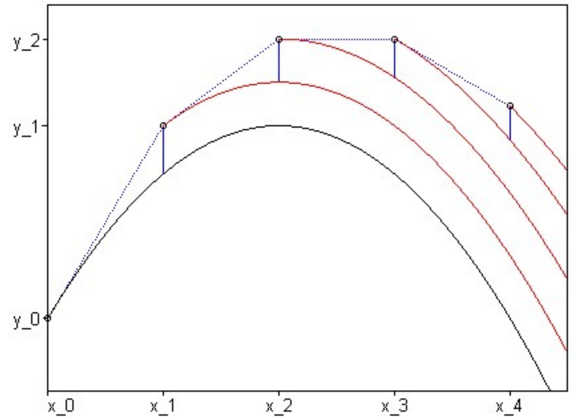


Method produces red line as approximation to the blue line

More generally: $y_{n+1} = y_n + h \cdot f(x_n,y_n)$, $n \geq 0$, with step size h.

Local and Cumulative Errors

Local error is the amount to which each tangent line departs from the soln curve between each successive (x_n, y_n) and (x_{n+1}, y_{n+1}).



If you add all of these individual errors up along the way (adding the lengths of the blue lines in the graph above), the total is the **cumulative error**.

Minimizing error: shorten step size until desired accuracy is achieved.
Limitations:

- Each time you divide the step size in half, you double the number of calculations, and thereby lengthen the time it takes the computer to calculate. Desired accuracy may take thousands of years!
- Computers round off each calculation to some decimal point. (this is error) The more calculations, the more error. Precious accuracy is slowly lost, and with small enough step size you may actually get less accuracy instead of more accuracy!

Which step size is best for Euler Method accuracy?

Complexities involved in answering this require a separate course!
However, graphing results of different step sizes combined with intuition regarding your function is often good enough.

What if you absolutely need more accuracy?

Improved methods are presented in subsequent sections of the book.

Also: If your unknown function y has a singularity, or is undefined in the domain you are analyzing, Euler method will produce unreliable results there.

Example: Solutions w/singularity: Given $y' = \frac{1}{x^2}$ with $y(-1) = 1$, what is $y(0)$ with step size $h = 0.2$?

For this simple DEQ, we can solve: $y = -\frac{1}{x} + C \quad \Rightarrow \quad 1 = -\frac{1}{-1} + C \quad \Rightarrow \quad C = 0.$

$\Rightarrow \quad y = -\frac{1}{x}$. So $y(0)$ is undefined, and $y \to \infty$ as $x \to 0^-$.

With Euler,

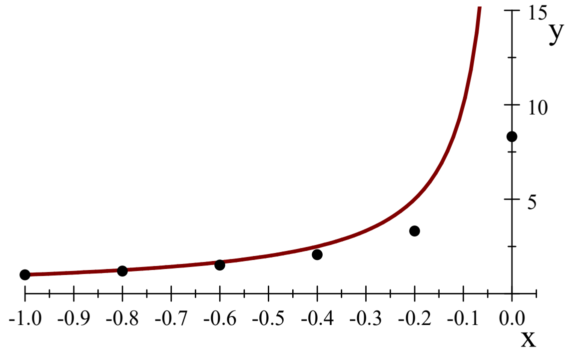
$$y_1 = y_0 + h \cdot f(x_0) = 1 + 0.2\left(\frac{1}{(-1)^2}\right) = 1.2$$

$$y_2 = y_1 + h \cdot f(x_1) = 1.2 + 0.2\left(\frac{1}{(-0.8)^2}\right) = 1.5125$$

$$y_3 = 1.5125 + 0.2\left(\frac{1}{(-0.6)^2}\right) = 2.0681$$

$$y_4 = 2.0681 + 0.2\left(\frac{1}{(-0.4)^2}\right) = 3.3181$$

$$y_5 = 3.3181 + 0.2\left(\frac{1}{(-0.2)^2}\right) = 8.3181 \stackrel{?}{=} \infty \qquad ?!?!$$



Euler method became unreliable as we approached the singularity at $x = 0$.

Exercises 2.4 

What did we learn?

- Euler Method
- Local and Cumulative Errors
- Limitations of Euler Method