2.3: Acceleration-Velocity Models

Recall that (when neglecting air resistance) the force ($F = ma$) exerted on a mass under gravity can be notated: $F_G = m\frac{dv}{dt} = -mg$ (where $g \approx 9.8\ m/\sec^2 \approx 32\ ft/\sec^2$ represents the acceleration of an object under the force of gravity at the surface of the Earth).

In real life, we have air resistance as an additional force, so to get the total force F_{total}, we must add in the additional air resistance force term F_R, so: $F_{total} = F_G + F_R$.

Due to various complexities (shape of the object, temperature, humidity, etc.), the force of air resistance (which also depends on velocity v) is written to allow us some flexibility: $F_R = \pm kv^p$ where k and p are positive constants with $1 \le p \le 2$. It suffices for our purposes to study instances when $p = 1$ or 2. So, $F_{total} = -mg \pm kv^p$.

Case: $p = 1$

Also known as "air resistance is proportional to the velocity."

$F_R = -kv = -k\frac{dy}{dt}$, where y is the distance from the ground.

Note that whether an object is falling or rising, the force of air resistance (F_R) has an opposite sign compared to the velocity ($\frac{dy}{dt}$).

As things rise, air resistance pushes down ($\frac{dy}{dt} > 0$ points up. So $F_R < 0$ points down, same as gravity).

And as things fall, air-resistance pushes up ($\frac{dy}{dt} < 0$, so $F_R > 0$, opposite from gravity).

So, $F_{tot} = F_G + F_R = -mg - kv$.

(confirm in your mind that $-kv$ is a positive number if an object is falling, but negative when rising)

Recalling $F = ma$, and therefore $a = \frac{F}{m}$, then our related accelerations from above are:

$$a_{tot} = \frac{F_G}{m} + \frac{F_R}{m} = a_G + a_R = -g - \frac{k}{m}v.$$

Definition: $\rho = \frac{k}{m}$ is the **drag coefficient**.

So in general (even when $p \neq 1$), we have $-\rho v^p$ is **acceleration due to air resistance**.

In the presence of air resistance, when falling, do you always accelerate?

$a_{tot} = 0$ when $-\rho v = g \quad \Rightarrow \quad v = -\frac{g}{\rho}$.

And $|v_\tau| := \frac{g}{\rho}$ is called the **terminal speed for** $p = 1$.

Case: $p = 2$

Also known as "air resistance is proportional to the **square** of the velocity ($a_R = \pm\rho v^2$)."

Recalling that we have $m\frac{dv}{dt} = -mg + F_R$, and that we need to keep the sign of air resistance (F_R) opposite the direction of velocity v, we get:

$F_R = -kv|v|$, and $a_{total} = \frac{dv}{dt} = -g - \rho v|v|$.

Upward Motion: happens when $v > 0$:

$$\frac{dv}{dt} = -g - \rho v^2 \qquad \text{(Eq. 12)}$$

Solving for:

Velocity: $$v(t) = \sqrt{\frac{g}{\rho}}\tan\left(\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right), \qquad \text{(Eq. 13)}$$

Position: $$y(t) = y_0 + \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)}{\cos\left(\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)}\right|. \qquad \text{(Eq. 14)}$$

Downward Motion: $v < 0$...

$$\frac{dv}{dt} = -g + \rho v^2 \qquad \text{(Eq. 15)}$$

Solving for:

Velocity: $$v(t) = \sqrt{\frac{g}{\rho}}\tanh\left(\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right), \qquad \text{(Eq. 16)}$$

Position: $$y(t) = y_0 - \frac{1}{\rho}\ln\left|\frac{\cosh\left(\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)}{\cosh\left(\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)}\right|. \qquad \text{(Eq. 17)}$$

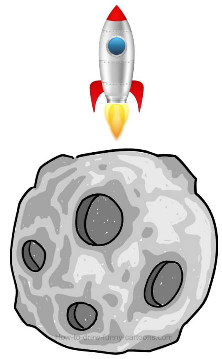
Note: $|v_\tau| = \sqrt{\frac{g}{\rho}} = \sqrt{\frac{mg}{k}}$ is the **terminal speed for** $p = 2$.

Universal Gravitational Force:

Assume two bodies have masses M and m, and are r distance apart.

Then, between them we have the force:

$F_G = \frac{GMm}{r^2}$, where $G \approx 6.6726 \times 10^{-11}\ N \cdot (m/kg)^2$.



Applying a constant thrust (acceleration) a_T to slow down the landing of a spacecraft…

$a = \frac{d^2r}{dt^2} = a_T - \frac{F_G}{m} = a_T - \frac{GM}{r^2}$, where M is the mass of the planet/moon,

m is the mass of the spacecraft, and r is the radius to the center of the planet.

$\frac{d^2r}{dt^2} = a_T - \frac{GM}{r^2}$ is a special type of 2nd order DEQ which allows a simplification to help solve it.

We can convert the DEQ into a 1st order DEQ (that we can solve) by using the chain rule,

(this works because $a_T - \frac{GM}{r^2}$ does not depend explicitly on t).

Observe that: $\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{dr}{dt} = \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr}v$ (notice the use of the chain rule).

So, $\frac{d^2r}{dt^2} = a_T - \frac{GM}{r^2}$ can be changed into: $v\frac{dv}{dr} = a_T - \frac{GM}{r^2}$.

Solving using separation of variables, we get:

$\frac{1}{2}v^2 = a_T r + \frac{GM}{r} + C$, (try calculating this yourself.)

We can now plug in our various given quantities (a_T, G, M) and any initial conditions ($v(r_0)$)

to determine our arbitrary constant C.

In particular, if we want the craft to land safely, perhaps we want to $v(R) = 0$ to be our initial condition.

Escape Velocity:

$v_0 = \sqrt{\frac{2GM}{r}}$ is the **escape velocity** on any planet/moon,

where r is distance from the center of the planet/moon to its surface, and M is the mass of the planet/moon.

To derive this, we start with Newton's gravitational force equation $F_G = \frac{GMm}{r^2}$.

So acceleration is: $a_G = \frac{d^2r}{dt} = \frac{GM}{r^2}$.

Using the chain rule as above: $\frac{d^2r}{dt} = \frac{dv}{dr}v = \frac{GM}{r^2}$.

Separating and then integrating: $\int v dv = \int \frac{GM}{r^2} dr \quad \Rightarrow \quad \frac{1}{2}v^2 = -\frac{GM}{r} + C$.

When $t = 0$, we label $R := r(0)$ and $v_0 := v(0)$, and substituting these in above we have:

$\frac{1}{2}v_0^2 + \frac{GM}{R} = C$, and therefore $v^2 = -2\frac{GM}{r} + v_0^2 + 2\frac{GM}{R} = v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R})$ [*]

Observe that for us to have escape velocity, the velocity must always be positive. In other words, even as $r \to \infty$.

Letting $r \to \infty$ in [*], we need $v^2 > v_0^2 - 2GM\frac{1}{R}$, where $v_0 \geq \sqrt{\frac{2GM}{R}}$.

And this is how (minimum) escape velocity ($v_0 = \sqrt{\frac{2GM}{r}}$) is defined.

Exercises

Problem: **#3** Suppose that a motorboat is moving at 40 *ft*/sec when its motor suddenly quits, and then 10 sec later the boat has slowed to 20 *ft*/sec. Assume, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

Eventually, what do we want to calculate?

$x(\infty) := x(t)$ as $t \to \infty$.

What is the equation for "resistance being proportional to velocity?"

$F_R = -kv$.

Dividing by mass, we have: $a = v' = -\rho v$.

How should we write the initial conditions?

$v(0) = 40; \qquad v(10) = 20, \qquad x(0) = 0.$

$\int \frac{1}{v} dv = -\rho \int dt, \qquad \ln|v| = -\rho t + c, \qquad v = e^{-\rho t + c} = e^c e^{-\rho t}$, where $e^c > 0$.

$40 = e^c, \qquad v(t) = 40e^{-\rho t},$

$20 = 40e^{-10\rho}, \qquad e^{-10\rho} = \frac{1}{2}, \qquad -10\rho = \ln\frac{1}{2} = -\ln 2, \qquad \rho = \frac{\ln 2}{10}.$

$\int dx = 40\int e^{-\rho t}dt, \qquad x = -\frac{40}{\rho}e^{-\rho t} + C,$

$0 = 40\left(-\frac{1}{\rho}\right)e^0 + C, \qquad C = \frac{40}{\rho}.$

$x(t) = \frac{40}{\rho}(1 - e^{-\rho t})$

"How far will the boat coast in all?"

$x(\infty) = \lim_{t\to\infty} \frac{40}{\rho}(1 - e^{-\rho t}) = \frac{40}{\rho} \approx 577\ ft.$

Problem: #17 Consider a crossbow bolt (from the example in the book) shot straight upward from the ground ($y = 0$), at time $t = 0$ with initial velocity $v_0 = 49\ m/s$. Use $g = 9.8\ m/s^2$ and $\rho = 0.0011$ in Eq. (12). Then show that the bolt reaches its maximum height of about $108.47\ m$ in about $4.61\ s$.

Recall:

$\frac{dv}{dt} = -g - \rho v^2$ (Eq. 12)

$\frac{dv}{dt} = -9.8 - 0.0011v^2$

$\int \frac{dv}{9.8+0.0011v^2} = -\int dt \qquad (*)$

Recall that: $(\tan^{-1}v)' = \frac{1}{1+v^2}.$

So the left-hand side of $(*)$ should change to:

$\int \frac{\frac{1}{9.8}}{1+(0.010595v)^2}dv = \frac{1}{(9.8)}\int \frac{1}{1+(0.010595v)^2}dv.$

u-sub: $u = 0.010595v, \qquad du = 0.010595dv$

$\frac{1}{9.8(0.010595)}\int \frac{1}{1+u^2}du = \frac{1}{9.8(0.010595)}\tan^{-1}(0.010595v) + C$

$\Rightarrow \tan^{-1}(0.010595v) = -0.103827t + C.$

$v(0) = 49$ implies $C + 0 = \tan^{-1}(0.010595 \cdot 49) = 0.478854.$

So, $\tan^{-1}(0.010595v) = -0.103827t + 0.478854$

$\Rightarrow 0.010595v = \tan(0.478854 - 0.103827t),$

$\Rightarrow v(t) = 94.3841\tan(0.478854 - 0.103827t).$

Or, if we just plug in our known values into Equation 13:

Velocity: $v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)$, we find we get the same answer.

"**Show that the bolt reaches its maximum height of about** $108.47\ m$ **in about** $4.61\ s$."

$$\int dy = 94.3841 \int \tan(0.478854 - 0.103827t)dt$$

$$\Rightarrow\ y = 94.3841\left[\tfrac{1}{0.103827} \ln|\cos(0.478854 - 0.103827t)|\right] + C$$

Plugging in our initial condition $y = 0$ at $t = 0$:

$$0 = 94.3841\left[\tfrac{1}{0.103827} \ln(\cos(0.478854))\right] + C$$

$$\Rightarrow\ C \approx 108.5.$$

So, $y(t) \approx 94.38[909 \ln|\cos(0.4788 - 0.1038t)|] + 108.5$

How do we find out the time the bolt is at its maximum height?

Recall from above: $v(t) = 94.38 \tan(0.4788 - 0.1038t)$

$0 = 94.38 \tan(0.4788 - 0.1038t)$, $\qquad 0.4788 - 0.1038t = \tan^{-1}(0) = 0$

$$\Rightarrow\ 0.4788 - 0.1038t = 0,$$

$$\Rightarrow\ t = \tfrac{0.4788}{0.1038} \approx 4.61 \text{ sec.}$$

So maximum height is: $y(4.61) \approx 94.38[909 \ln|\cos(0.4788 - (0.1038)(4.61))|] + 108.5$

$$\approx 108.47\ m.$$

Problem: **#22**

Suppose that $\rho = 0.075$ in Eq. 15 ($\frac{dv}{dt} = -g + \rho v^2$ in fps units, with $g = 32\ ft/s^2$) for a paratrooper falling with parachute open. If he jumps from an altitude of $10{,}000$ feet and opens his parachute immediately, what will be his terminal speed? How long will it take him to reach the ground?

By an integration (or Eq. 13: $v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)$), the solution of the initial value problem $v' = -32 + 0.075v^2$, $v(0) = 0$ is $v(t) = -20.666 \tanh(1.54919t)$.

And since $\tanh(t) \to -1$ as $t \to -\infty$, we have:

terminal speed $= |v_\tau| = \left|-\sqrt{\frac{g}{\rho}}\right| = 20.666\ ft/\text{sec}$.

"**How long will it take him to reach the ground**?"

$$\text{Position: } y(t) = y_0 - \tfrac{1}{\rho} \ln\left|\frac{\cosh\left(\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)}{\cosh\left(\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)}\right|. \qquad \text{(Eq. 17)}$$

With $y(0) = 10,000$, we get $y(t) = 10,000 - 13.333\ln(\cosh(1.54919t))$.

We solve for $y(t) = 0$, and get $t = 484.57$ sec.

$\frac{484.57\,\text{sec}}{60\,\text{min/sec}} \approx 8.0762\,\text{min}$. And $0.0762\,\text{min} \cdot 60\frac{\text{sec}}{\text{min}} \approx 4.57\,\text{sec}$.

Thus the descent takes about 8 min 5 sec.

Problem: #**24**

The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the Earth.

a) To what radius would the earth have to be compressed in order for it to become a black hole? The escape velocity from its surface would need to be equal to $3 \times 10^8\ m/s$ (the speed of light!).

Recall: Escape Velocity is $v_0 = \sqrt{\frac{2GM}{R}}$.

Also recall: that the gravitational constant $G = 6.6726 \times 10^{-11}\ (kg \cdot m \cdot s^2) \cdot (m/kg)^2$, and the mass of the earth is $M = 5.975 \times 10^{24}\ kg$.

So, $\sqrt{\frac{2GM}{R}} = 3 \times 10^8\ m/s$, and

$$R = \frac{2GM}{(3\times10^8\ m/s)^2} = \frac{2\left(6.6726\times10^{-11}\ (kg\cdot m\cdot s^2)\cdot(m/kg)^2\right)\left(5.975\times10^{24}\ (kg)\right)}{(3\times10^8\ m/s)^2}$$

$R = \frac{79.6975344\times10^{13}}{9\times10^{16}} = 8.85973 \times 10^{-3}$ meters, or about $0.88\ cm$.

b) Repeat part a) with the sun in place of the earth.

So, $\sqrt{\frac{2G(329,320\cdot M)}{R}} = c$, then $R = \frac{2\cdot329,320\cdot GM}{(3\times10^8\ m/s)^2} = \frac{658,320\left(6.6726\times10^{-11}\ (kg\cdot m\cdot s^2)\cdot(m/kg)^2\right)\left(5.975\times10^{24}\ (kg)\right)}{(3\times10^8\ m/s)^2}$

$R = \frac{26246418.5412\times10^{13}}{9\times10^{16}} = 2916268.7268 \times 10^{-3}$ meters, or about $2.92\ km$ (1.8 miles!!).