4/25/2026

Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

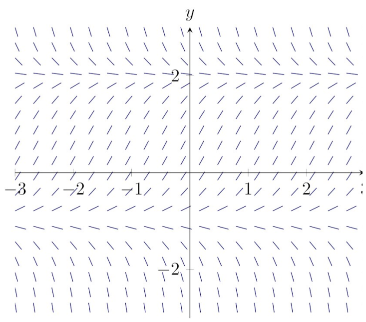
Previous Lecture

- Population models with variable birth/death rates
- Logistic Equation: $\frac{dP}{dt} = kP(M - P)$
- Partial Fractions: $\frac{1}{f(x)g(x)} \stackrel{?}{=} \frac{A}{f(x)} + \frac{B}{g(x)}$

2.2: Equilibrium Solutions and Stability

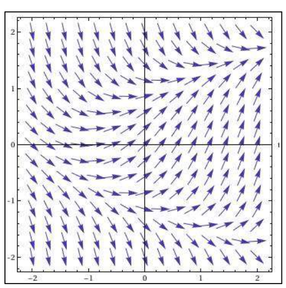
In this section, we study **Autonomous DEQs** $\frac{dy}{dt} = f(y)$ whose slope fields *do not* change w/time (w/their independent variable).
Observe $f(y)$ does not *explicitly* depend on t (or it's independent variable).

For example: $\frac{dy}{dt} = f(y) := y(t) + a$ or $\frac{dy}{dx} = f(x) := y(x)^2 - by(x) + y(x)$, etc.

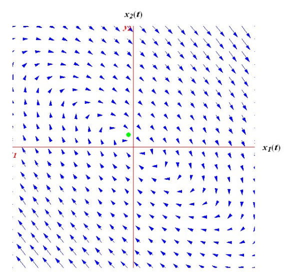


slope depends on y (vert), not t (horz).

Alternatively, non-autonomous DEQs *do* depend explicitly on t, for example: $\frac{dy}{dt} = f(y, t) := y(t) + at^2 + b$.



1D Non-Autonomous



2D Non-Autonomous (animated in class)

If we can't solve a DEQ, then besides its slope field, what else can we figure out?

Let's assume we're dealing w/an autonomous DEQ. (because even if we aren't, there's a trick to convert the DEQ into one!)

Critical points (of an autonomous DEQ $y' = f(y)$): These are sols to the equation $f(y) = 0$ (i.e., y_0 such that $f(y_0) = 0$).
These are pts y_0 where the slope $f(y_0)$ is zero, so the value of the function y is not changing.

These pts are very important, because they divide up our domain into regions w/qualitatively different behaviors.

Examples of critical pts in the figure below are the values $y_0 = -2$ and $y_0 = 3$.

Simple Diagram: $\uparrow$ 3 $\downarrow$ -2 $\uparrow$

OR, w/time axis:

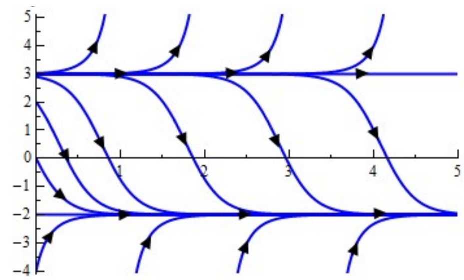


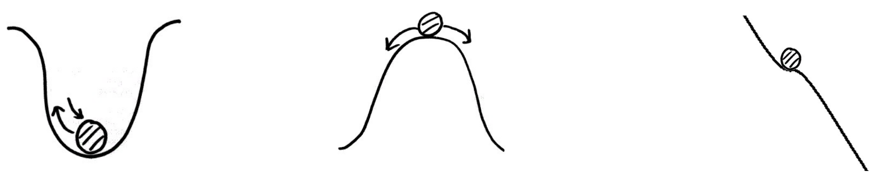
Fig 1: 1D Autonomous: $y' = f(y) = y^2 - y - 6$

Equilibrium Solution: If y_0 is a critical pt, then the DEQ has the constant soln $y(t) = y_0$, called an equilibrium soln. Examples of equilibrium sols in the above figure are the functions $y(t) \equiv -2$ and $y(t) \equiv 3$.

Stable/Unstable Critical Points

A critical pt $y = c$ of an autonomous DEQ is **stable** if, for all init-vals y_0 of the soln $y(t)$ sufficiently close to c, $y(t)$ remains close to c, for all $t > 0$ (recall that $y_0 := y(0)$).

Otherwise, the critical pt c is considered to be **unstable**. Unstable means that no matter how close to c your init-val y_0 is, the soln $y(t)$ may drift away from c, for some $t > 0$.



Stable | Unstable | Unstable (Semi-Stable) Equilibrium

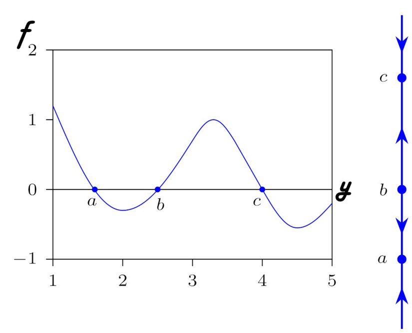
How do we discern stability? (and soln behavior generally?)

Phase Diagram: A type of diagram used to distinguish between qualitatively different parts of the domain. These parts are separated by equilibria. In Fig. 1, the phase diagram would be: $\rightarrow \ -2 \ \leftarrow \ 3 \ \rightarrow$

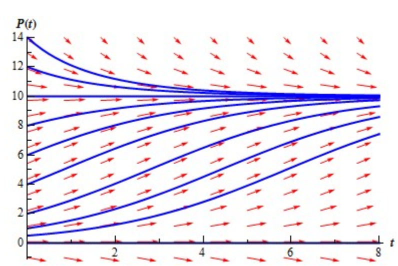
How to construct this?

- Find critical points
- Test points around critical points to see if $f > 0$ or $f < 0$.

1D

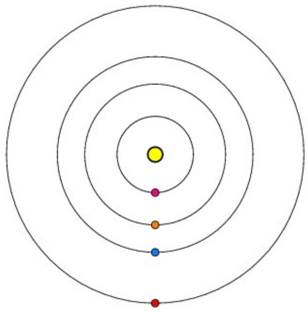


Types of critical points: 1D sinks/sources. 2D funnels/spouts/etc.

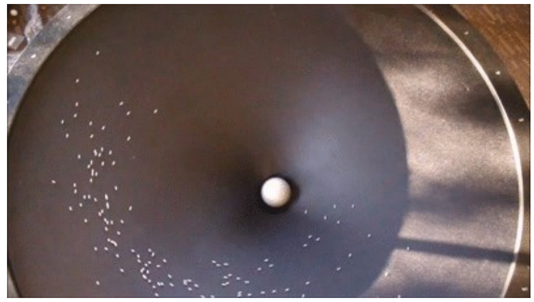


Logistic example of sink

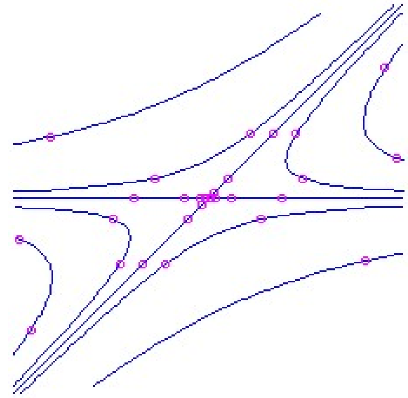
2D



Stable critical point (animated in class)



Stable critical point - funnel (animated in class)



Unstable critical point (animated in class)

More formally, a critical pt c is **stable** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$|y_0 - c| < \delta$ implies that $|y(t) - c| < \varepsilon$, for all $t > 0$. (you'll have to think thru this carefully)

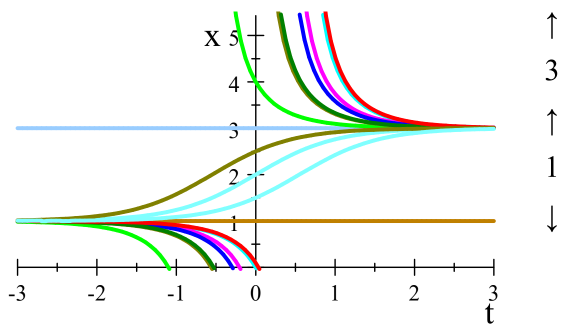
The critical point $y = c$ is **unstable** if it's not stable.

Bifurcation Point: Given a DEQ with parameter h (for example, $\frac{dx}{dt} = x(4-x) - h$), a bifurcation pt is a pt $h = h_0$ where the number of equilibrium pts for the DEQ changes depending upon whether h is greater or less than h_0.

For $x' = x(4-x) - h$, note that when $h = 3$:

$$x(4-x) - 3 = -x^2 + 4x - 3 = -(x-3)(x-1) = 0$$

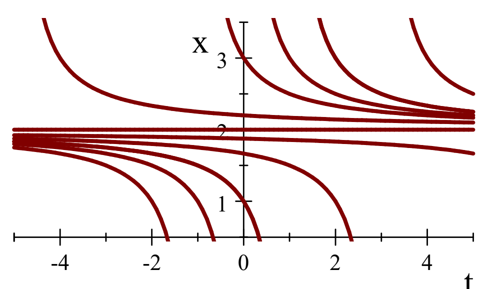
$\Rightarrow \; x \in \{3, 1\}$. Two equilibrium pts.



Soln is $\frac{3(C-1)-1(C-3)e^{-2t}}{(C-1)-(C-3)e^{-2t}}$, for various C

However, when $h = 4$, we have: $x(4-x) - 4 = 0$

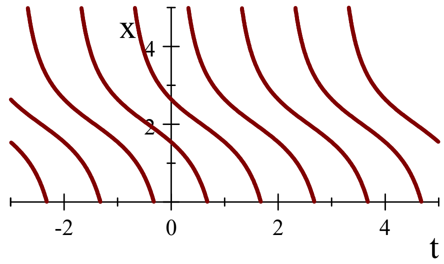
$\Rightarrow \; x^2 - 4x + 4 = (x-2)^2 = 0$ when $x = 2$. Only one critical pt!



$\frac{2t-2C+1}{t-C}$, for various C

And when $h = 5$, we have: $x(4-x) - 5 = -x^2 + 4x - 5 = 0$

$\Rightarrow x \in \{2 \pm i\}$. So no (real) critical pts.



$x = 2 - \tan\left(t + C - \frac{\pi}{2}\right)$, for various C

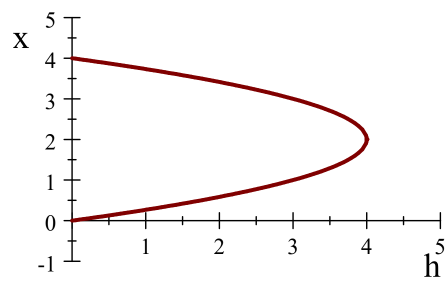
So more generally, solving $x(4-x) - h = 0$,

$$x^2 - 4x + h = 0$$

$$\Rightarrow \quad x(h) = \frac{4 \pm \sqrt{16 - 4(1)(h)}}{2} = 2 \pm \sqrt{4 - h}.$$

Two critical pts when $h < 4$. Therefore, there's a bifurcation pt at $h_0 = 4$.

Bifurcation Diagram: A diagram in the hx-plane (see below) which visualizes the bifurcation of the critical pts ($f = 0$) as you vary the bifurcation parameter h.



Mapping critical points $x(h) = 2 \pm \sqrt{4-h}$ in the hx-plane.

Exercises 2.2 

What did we learn?

- Autonomous DEQ
- Critical Pts/Equilibrium Sols
- Stable/Unstable Critical Pts
- Bifurcation Pts/Diagram