Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Integrating Factor
- Solving Linear 1st-Order DEQ

1.6: Substitution Methods and Exact DEQs

Substitution Method

How to solve a 1st-order DEQ which is **not separable or linear**? One way is through a *substitution* method, thereby transforming our DEQ into a form we know how to solve.

Sometimes we have a DEQ, $\frac{dy}{dx} = f(x,y)$ where $f(x,y)$ contains some expression $\alpha(x,y)$ (possibly occurring repeatedly) that would make the DEQ easier to solve if we substituted $\alpha(x,y)$ out w/a new variable $v := \alpha(x,y)$. (kinda like u-sub)

Example: $\frac{dy}{dx} = f(x,y) = (x+y+3)^2$ (note this is non-linear)

Obviously this would be simpler to solve if we substituted out $x+y+3$ with $v := x+y+3$.

But if we make the substitution on the RHS, what does the LHS look like?

The LHS has us taking the derivative of y, so I should first write y in terms of v.

Notice that we can rewrite our substitution as: $y = \beta(x,v) := v - x - 3$.

So, using the chain rule on this multivariable $\beta(x,v)$:

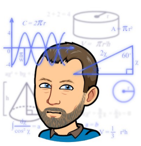
$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x}\frac{dx}{dx} + \frac{\partial\beta}{\partial v}\frac{dv}{dx} = -1 + (1)\frac{dv}{dx} = \frac{dv}{dx} - 1.$$ (total derivative from multivar calc)

Substituting this into the LHS of our DEQ, we have:

$$\frac{dv}{dx} - 1 = v^2 \quad \text{or} \quad \frac{dv}{dx} = v^2 + 1.$$ (seperable!)

Or, following this process for some other DEQ you will find more generally: $\frac{dv}{dx} = g(x,v)$. (RHS might be a function of v *and* x)

If you find a solution $v(x)$ to this DEQ, then $y = \beta(x, v(x))$ will be a solution of the original DEQ.



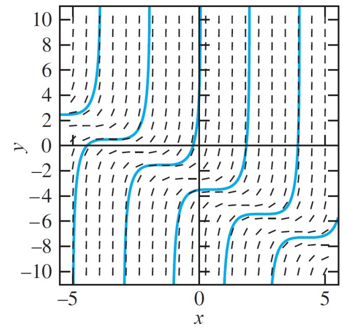
Often, this process may require some ingenuity or trial and error.

Back to our example: using separation of variables: $\frac{1}{v^2+1}\frac{dv}{dx} = 1 \quad \Rightarrow \quad \int \frac{1}{v^2+1} dv = \int dx$

So, $x = \tan^{-1} v + C$ or $v = \tan(x - C)$. (but remember, our original DEQ $\frac{dy}{dx}$ requires a solution $y = f(x)$)

Note: because $v := x + y + 3$, we have $x + y + 3 = \tan(x - C)$ or $y = \tan(x - C) - x - 3$. ☐

❗ Recall that *linear* DEQs w/cont. coefficient functions always have cont. sols. This isn't always true for *non-linear* DEQs.
For instance, note that $\tan x$ is cont. on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but has periodic discontinuities. And in this case, the constant means that the solution is cont. on $-\frac{\pi}{2} < x - C < \frac{\pi}{2}$ or $C - \frac{\pi}{2} < x < C + \frac{\pi}{2}$.



Discontinuous solns for

The substitution technique used in the example above can be applied to any DEQ of the form: $\frac{dy}{dx} = F(ax + by + c)$.

But what about solving other classes of 1st-order DEQs?

1st-Order DEQs with Scalar Homogeneity

A **1st-order DEQ with scalar homogeneity** is one that can be written in the form: $\frac{dy}{dx} = F(\frac{y}{x})$. (e.g., $y' = 7\frac{x}{y} + 9\frac{y}{x}$)

❗ Note that this is a different use of the word *homogeneous*.
Here, scalar homogeneity does not refer to the linear homogeneity we saw earlier,
Linear homogeneity meant DEQs had a zero constant (in y, y') term.
Rather *scalar homogeneity* refers to the function on the RHS of the DEQ being written in the form $F(\frac{y}{x})$.

Note if we make the substitution: $v = \frac{y}{x}$, or equivalently $y = vx$, we have: $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Or equivalently: $F(v) = v + x\frac{dv}{dx}$.

We have thus transformed this into a separable DEQ: $x\frac{dv}{dx} = F(v) - v$.

Therefore, *every* 1st-order DEQ with scalar homogeneity can be reduced to an integration problem.

❗ Sometimes DEQs can be easily manipulated to be in this form. In general, if your DEQ is of the form
$A(x,y)\frac{dy}{dx} = B(x,y)$,
you can attempt to divide by $A(x,y)$, and then check to see if you can manipulate it to be in the correct form.

Example: Solve: $2xy\frac{dy}{dx} = 4x^2 + 3y^2$ ...

$$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x} \qquad \text{(when } x,y \neq 0\text{)}$$

$$v = \frac{y}{x}, \quad y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}, \quad \text{and} \quad \frac{1}{v} = \frac{x}{y}.$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v$$

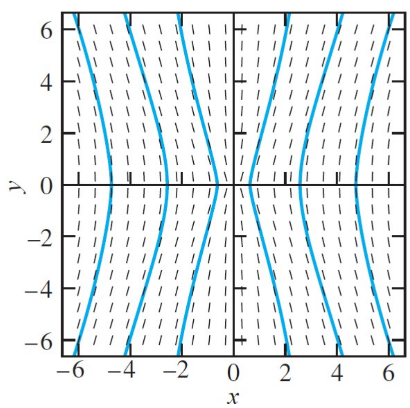
$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2}{v} + \frac{v}{2} = \frac{v^2+4}{2v}$$

$$\Rightarrow \quad \int \frac{2v}{v^2+4}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad \ln(v^2+4) = \ln|x| + \ln C$$

$$\Rightarrow \quad v^2 + 4 = C|x| \quad \Rightarrow \quad \frac{y^2}{x^2} + 4 = C|x| \quad \Rightarrow \quad y^2 + 4x^2 = C|x^3|.$$

$$\Rightarrow \quad y = \pm\sqrt{C|x^3| - 4x^2}.$$

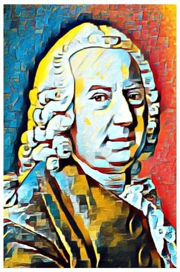


$\pm\sqrt{C|x^3| - 4x^2}$

Bernoulli DEQs

A Bernoulli is a non-linear 1st-order DEQ of the form:

$$\frac{dy}{dx} + yP(x) = y^nQ(x). \qquad (n \in \mathbb{R} \text{ and } n \neq 0, 1)$$



Bernoulli

Applications: population w/crowding, spread of disease, chemical reactions, electrical circuits w/nonlinear resistance, fluid mechanics w/drag forces, economicic growth w/saturation, etc.

To solve, we first define: $v := y^{1-n} = \frac{y}{y^n}$, this can transform the DEQ.

Taking a derivative of v, we find: $\frac{dv}{dx} = \frac{d}{dx}y^{1-n} = (1-n)y^{-n}\frac{dy}{dx}$.

Rearranging this, we find: $\frac{dy}{dx} = \frac{y^n}{(1-n)}\frac{dv}{dx} = \frac{y}{(1-n)v}\frac{dv}{dx}$.

Substituting this into our Bernoulli (attempting to convert to a $v(x)$ DEQ): $\frac{y}{(1-n)v}\frac{dv}{dx} + yP(x) = \frac{1}{v}yQ(x)$

OR (equivalently) $\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$. (wonderful! It's 1st-order linear.)

Don't bother to memorize this last expression. Instead, memorize the substitution $v := y^{1-n}$, then do the substitution. Let's do an example.

Example: $2xy\frac{dy}{dx} = 4x^2 + 3y^2$.

It's not Bernoulli yet, but what if we isolate the derivative by dividing by $2xy$:

$\frac{dy}{dx} - \frac{3}{2x}y = 2x\frac{1}{y}$ (when $x, y \neq 0$) $(*)$

We now have the Bernoulli form with $P(x) = -\frac{3}{2x}$, $Q(x) = 2x$, and $n = -1$ or $1 - n = 2$.

Thus: $v = y^{1-n} = y^2$ or $y = \pm v^{\frac{1}{2}}$. Taking a derivative: $\frac{dy}{dx} = \pm\frac{dy}{dv}\frac{dv}{dx} = \pm\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$. (chain rule)

Substituting into $(*)$: $\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} - \frac{3}{2x}v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}$.

Multiplying by $2v^{\frac{1}{2}}$ (to put it in normal form) produces the linear equation $\frac{dv}{dx} - \frac{3}{x}v = 4x$??

with integrating factor $I(x) = e^{\int -\frac{3}{x}dx} = x^{-3}$. So: $D_x(x^{-3}v) = \frac{4}{x^2}$.

$\Rightarrow \quad x^{-3}v = -\frac{4}{x} + C$ (upon integrating)

$\Rightarrow \quad x^{-3}y^2 = -\frac{4}{x} + C$

$\Rightarrow \quad y^2 = x^2(Cx - 4)$. $\square$

Exact DEQs

Another option is to put your 1st-order non-linear DEQ in the following form: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, where M, N are just some expressions in x, y.

Now we will use the following fact:

All 1st-order DEQs (where the coefficient functions have cont. first derivatives) have a (local) solution taking the implicit form: $F(x, y(x)) = C$ (even if we can't solve for it).

So, assuming F exists, we take a derivative wrt x, giving: $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$.

Note that this has the same form as our DEQ above ($\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are just expressions in x,y like M,N are).

So, if we can find an expression $F(x,y(x))$ w/the "exact" properties of: $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$,
then this will be an implicit soln to the original DEQ! Namely $F = C$ where C determines the particular sol.

DEQs that have a solution F w/these 'exact' properties are called **Exact DEQs**. But not all 1st-order DEQs are exact.

Existence Criteria for F

Luckily, there are *criteria* by which we can *know* if an F w/exact properties exists!

Recall from multivar calculus, that for any $F(x,y)$, if the mixed 2nd-order partial derivatives $F_{xy} := \frac{\partial^2 F}{\partial x \partial y}$ and $F_{yx} := \frac{\partial^2 F}{\partial y \partial x}$ are continuous on an open set in the xy-plane, then they are equal: $\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x}$. (Clairaut's Thm)

Therefore, if an F w/exact properties exists (i.e., if the DEQ is exact), and has cont. 2nd derivatives, then it must be true that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

This is easily checked. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then there is no point in searching for such an F, because the DEQ is not exact.

Example: Solve $y^3 dx + 3xy^2 dy = 0$.

We can rewrite this as $y^3 + 3xy^2 \frac{dy}{dx} = 0$, where $M = y^3$ and $N = 3xy^2$.

Note that $M_y = 3y^2$ and $N_x = 3y^2$, so it might be exact!

Okay, if it is exact, we should be able to integrate M wrt x, and discover F.

Note: $F(x,y) = \int M dx = xy^3 + g(y)$ ($g(y)$ is constant wrt x)

But what is $g(y)$? Well, recall that F_y is $N = 3xy^2$. So, taking a derivative of our F in the previous line wrt y:
$F_y = 3xy^2 + g'(y)$. So, it must be that $g'(y) = 0$, so $g(y)$ must be a constant wrt y as well.

Thus, our solution is: $F(x,y) = xy^3 + C = 0$. □

Formally:

Criteria for Exactness Thm: Suppose M,N are continous and have cont. 1st-order partial derivatives in the open rectangle $R : a < x < b,\ c < y < d$. Then the DEQ: $M + N\frac{dy}{dx} = 0$ is exact in R **iff** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ at each point of R.
That is, $\exists F$ on R with $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ **iff** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds on R.

[**Proof in book**]

Applications: consumer theory, groundwater flow, aerodynamics (flow around wings), gravitational potential contour lines in geography (constant altitude), isotherms in heat transfer, electrostatic potential equipotential curves in electrostatics, economics (utility functions), spring–mass systems, etc.

Procedure when presented w/a 1st-order DEQ:

- Put DEQ in exact form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
- Check whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- If it is, calculate $F(x,y) = \int M dx = f(x,y) + g(y)$.
- Next, calculate $F_y = \frac{\partial}{\partial y}(f(x,y) + g(y))$, and set it equal to $N(x,y)$.
- This equation should allow you to determine $g(y)$ and therefore the soln: $F(x,y) = C$.

Example: Solve: $2xy + 3 + (x^2 + 4y)\frac{dy}{dx} = 0$.

Here: $M = 2xy + 3$ and $N = x^2 + 4y$. ...

Note: $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$, therefore the DEQ is exact.

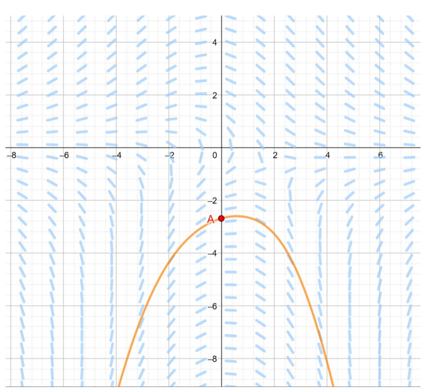
Integrating wrt x: $F(x,y) = \int(2xy + 3)dx = x^2y + 3x + g(y)$.

Also: $F_y = \frac{\partial}{\partial y}(x^2y + 3x + g(y)) = x^2 + g'(y) = x^2 + 4y$, therefore $g'(y) = 4y$ or $g(y) = 2y^2 + C$.

Therefore our implicit solution is: $F(x,y) = x^2y + 3x + 2y^2 + C = 0$.

❗ Where does a unique soln exist for the above DEQ? Rewriting it in "existence & uniqueness" thm form (see previous notes):

$\frac{dy}{dx} = f(x,y) = -\frac{2xy+3}{x^2+4y}$ and note $f_y = -\frac{2x^3-12}{(x^2+4y)^2}$. These are both continuous when $x^2 + 4y \neq 0$ (denom not zero), so not on the quadratic $y = -\frac{1}{4}x^2$. Everywhere else we have a unique solution.



Slope field for: $2xy + 3 + (x^2+4y)\frac{dy}{dx} =$

Exercises 1.6

What did we learn?

- 1st-Order DEQs Substitution Methods
- Scalar Homogeneity: $y' = F(\frac{y}{x})$
- Bernoulli DEQs: $y' + yP(x) = y^n Q(x)$
- Exact DEQs: $M(x,y) + N(x,y)y' = 0$

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