Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Separable DEQs
- Implicit Solutions

1.5: Linear 1st-Order DEQs

Integrating Factor

We previously solved separable DEQs like $\frac{dy}{dt} = 2ty$ with an integrating factor.

Note: this factor let both sides of the DEQ resemble a derivative.

For $\frac{1}{y}\frac{dy}{dt} = 2t$, the LHS is $\frac{d}{dt}(\ln y)$, and the RHS is $\frac{d}{dt}(t^2)$.

So we can integrate both sides. But, *non-separable* DEQs are more common.

Example: Newton's law of cooling when the ambient temperature is not constant:

$T'(t) = k(A(t) - T)$. How do we solve?

Linear 1st-Order DEQs

Are any of these **linear**? $\frac{dy}{dx} + y^2 + c = 0$, $\frac{dy}{dx} = y + \frac{x}{y} + c$, $\frac{dy}{dx} = y^{\frac{1}{5}}$.

How can we solve linear 1st-order DEQs more generally?
First, we need them in **normal form**: $\frac{dy}{dx} + P(x)y = Q(x)$, where P, Q are continuous.

Can't just integrate right away, and the integrating factor isn't obvious.

Turns out, the integrating factor is: $I(x) := e^{\int P(x)dx}$.

Example: $\frac{1}{2}xy' + y = x$ …

Normal form (for $x \neq 0$): $y' + \frac{2}{x}y = 2$. …

$e^{2\int \frac{1}{x}dx} = e^{2\ln|x|+c} = e^c x^2.$

$I(x) := x^2.$ (since we're multiplying both sides of DEQ by $I(x)$, we're able to cancel nonzero constant e^c)

Multiplying both sides of DEQ by $I(x)$: $x^2y' + 2xy = 2x^2.$

The Trick: Recognize the LHS of DEQ as the result of the product rule.

$x^2y' + 2xy = \frac{d}{dx}(x^2y) = \frac{d}{dx}(I(x)y).$

Therefore, DEQ can be rewritten as $\frac{d}{dx}(x^2y) = \frac{d}{dx}\left(\frac{2}{3}x^3\right).$

Integrating: $x^2y = \frac{2}{3}x^3 + C \quad \Rightarrow \quad y = \frac{2}{3}x + \frac{C}{x^2}.$

Justification for $e^{\int P(x)dx}$

Want to multiply by some $I(x)$ such that $\frac{dy}{dx} + P(x)y = Q(x)$ becomes integrable.

$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$ (*)

Note that the LHS might be integrable if it's the result of product rule: $\frac{d}{dx}[I(x)y(x)] = I(x)\frac{dy}{dx} + I'(x)y(x).$

Observe from LHS of (*), this would require $I'(x) = I(x)P(x)$ …

This is separable!

$\int \frac{1}{I(x)}dI = \int P(x)dx \quad \Rightarrow \quad \ln|I(x)| = \int P(x)dx \quad \Rightarrow \quad I(x) = e^{\int P(x)dx}.$ (Why not $\pm$? or $+c$?)

No singular solutions (because $I \neq 0$).

Linear 1st-Order DEQ Existence/Uniqueness Thm: If functions $P(x), Q(x)$ are continuous on an open interval containing x_0, then the init-val problem: $\frac{dy}{dx} + P(x)y = Q(x)$, with init-conds $y(x_0) = y_0$ has a unique soln $y(x)$.

Example: Find general solution of: $(x^2+4)y' + 3xy = x$. Then, using $y(0) = 1$, find the particular solution.

Correct form?

$$y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4} \qquad (x^2+4 \neq 0, \text{ for any } x)$$

$$I = \exp\left(\int \frac{3x}{x^2+4}dx\right),$$

$$u = x^2+4, \qquad du = 2xdx$$

$$I = \exp\left[\frac{3}{2}\int \frac{1}{u}du\right],$$

$$I = \exp\left[\frac{3}{2}\ln(x^2+4)\right],$$

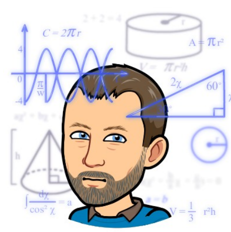
$$I = e^{\ln\left((x^2+4)^{\frac{3}{2}}\right)} = (x^2+4)^{\frac{3}{2}}. \qquad \text{Now what?}$$

Multiply both sides of DEQ ($y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$) by $I(x)$:

$$I(x)\left(y' + \frac{3x}{x^2+4}y\right) = I(x)\frac{x}{x^2+4} \qquad \text{or} \qquad (x^2+4)^{\frac{3}{2}}\left[y' + \frac{3x}{x^2+4}y\right] = (x^2+4)^{\frac{3}{2}}\left[\frac{x}{x^2+4}\right].$$

Simplifying, we have: $(x^2+4)^{\frac{3}{2}}y' + 3x(x^2+4)^{\frac{1}{2}}y = x(x^2+4)^{\frac{1}{2}}$. . . .

Next, *recognize* LHS of DEQ is derivative of: $(x^2+4)^{\frac{3}{2}}y$, which is $I(x)y$.



So, $(I(x)y)' = x(x^2+4)^{\frac{1}{2}}$

(remember our goal is to discover what y is!)

Integrating both sides:

LHS: $I(x)y$ or $y(x^2+4)^{\frac{3}{2}}$

RHS: $\int\left[x(x^2+4)^{\frac{1}{2}}\right]dx.$

Using u substitution: $\quad u = x^2+4, \quad du = 2xdx,$ we have:

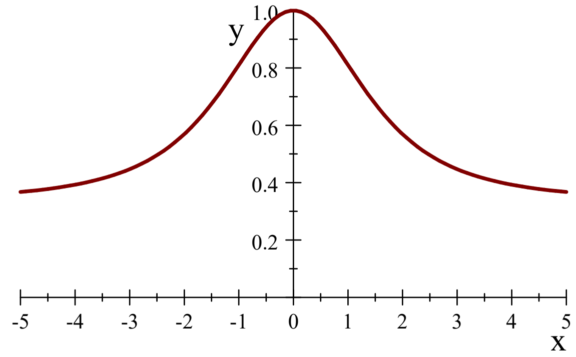
$$\int\left[x(x^2+4)^{\frac{1}{2}}\right]dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}+C\right) = \frac{1}{3}(x^2+4)^{\frac{3}{2}}+\frac{1}{2}C$$

$$y = \frac{1}{I(x)}\left(\frac{1}{3}(x^2+4)^{\frac{3}{2}}+\frac{1}{2}C\right) = \frac{1}{3}+\frac{1}{2}C(x^2+4)^{-\frac{3}{2}}$$

"Using $y(0) = 1$, find particular solution"

$$1 = \frac{1}{3}+\frac{1}{2}C(0^2+4)^{-\frac{3}{2}} = \frac{1}{3}+\frac{1}{16}C, \qquad \Rightarrow \qquad C = \frac{32}{3}.$$

So $y(x) = \frac{1}{3}+\frac{16}{3}(x^2+4)^{-\frac{3}{2}}$.



$\frac{1}{3}+\frac{16}{3}(x^2+4)^{-\frac{3}{2}}$

Applications of Linear 1st-Order DEQs

- Electrical circuits
- Newton's law of cooling (w/variable ambient temp)
- Problems involving mixing solutions in a container
- 1st-order chemical reactions
- Population models with harvesting (e.g., harvesting fish from a lake)

Exercises 1.5

What did we learn?

- Integrating Factor
- Solving Linear 1st-Order DEQ

Prepared by Dr. Jodin Morey. Materials for Other Courses Found at **MathTalker.org**