Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Direction/Slope Fields of DEQs
- Existence and Uniqueness of Sols Criteria

§1.4: Separable DEQs and Applications

Separation of Variables

Beyond graphing slope fields and estimates of solutions, can we exactly solve *ANY* 1st-order DEQs when y' is a function of both x AND y, like: $y' = f(x,y)$? (sometimes) For example: $y' = y(x^2+7)$.

Can't just integrate right away, but can we multiply both sides of DEQ by a factor which allows us to then integrate? (sometimes!)

When we can, this factor is called an **integrating factor**. It's called this because after multiplying, both sides of the DEQ take the form of a derivative, allowing us to integrate.

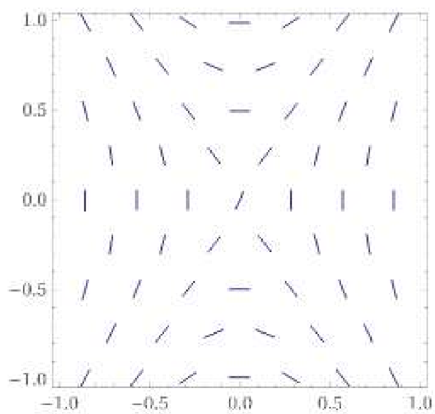
Example: $\dfrac{dy}{dx} = \dfrac{9x}{4y}$

$$4y\,dy = 9x\,dx \quad \text{or} \quad 4y\frac{dy}{dx} = 9x \qquad \text{(integrating factor } 4y\text{)}$$

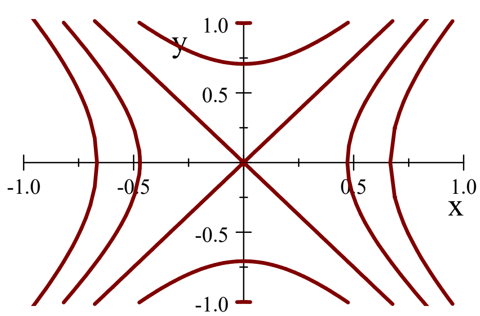
$$\int 4y\,dy = \int 9x\,dx \quad \Rightarrow \quad \int (2y^2)'\,dy = \int \left(\tfrac{9}{2}x^2\right)'\,dx$$

$$\Rightarrow 2y^2 = \tfrac{9}{2}x^2 + c \qquad \text{(implicit solution)}$$

$$y = \pm\sqrt{\tfrac{9}{4}x^2 + \tfrac{c}{2}}.$$



slope field



$y = \pm\sqrt{\tfrac{9}{4}x^2 + \tfrac{c}{2}}$ for various c

Example: $\frac{dy}{dx} = 2x(y+2)$

$\frac{1}{y+2}\frac{dy}{dx} = 2x$ when ??

$y(x) \neq -2.$ (case 1)

$\int \frac{1}{y+2}dy = \int 2x dx$

$\ln|y+2| = x^2 + c,$

$e^{\ln|y+2|} = e^{x^2+c}$

$y + 2 = \pm e^{x^2}e^c$

$y + 2 = de^{x^2}$, where $d \neq 0$.

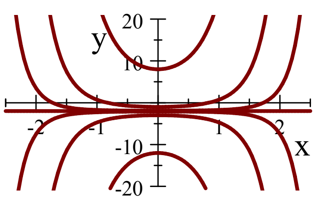
So our (partial) solution is: $y = de^{x^2} - 2$, where $d \neq 0$.

But what happens if $y(x) \equiv -2$? (case 2)

Observe that for both sides of the given DEQ, we have:

$\frac{dy}{dx} = \frac{d(-2)}{dx} = 0$ and $2x(y+2) = 2x(-2+2) = 0$.

So, $y(x) \equiv -2$ *IS* a solution. The FULL solution is: $y = de^{x^2} - 2$, for any $d \in \mathbb{R}$ (d can be zero!).



$y = de^{x^2} - 2$

More Formally

Separable DEQ: If you can (re)write $\frac{dy}{dx} = f(x,y)$ as $\frac{dy}{dx} = g(x)h(y)$ (an expression in x multiplied by an expression in y), with $h(y) \neq 0$, then rearrange the DEQ as: $\frac{1}{h(y)}\frac{dy}{dx} = g(x)$. (integrating factor $\frac{1}{h(y)}$)

Now you hope to integrate both sides of the DEQ to get an implicit soln:

$$\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx. \qquad (*)$$

Which of the following are separable? $y' = x^2y + y\sin x$ $y' = x^2 + y\sin x$

Avoiding the Shortcut of using $\frac{dy}{dx}$ as a fraction (optional)

We used a shortcut previously (treating $\frac{dy}{dx}$ as a fraction), but more formally we define a function:

$H(y) := \int \frac{1}{h(y)} dy$, and note that $H'(y) = \frac{1}{h(y(x))}$.

Then, by the chain rule, $\frac{d}{dx} H(y(x)) = H'(y(x)) \frac{dy}{dx} = \frac{1}{h(y(x))} \frac{dy}{dx}$.

So, going back to the LHS of $(*)$, we have $\int \frac{1}{h(y(x))} \frac{dy}{dx} dx = \int \frac{d}{dx} H(y(x)) dx = H(y(x)) + C = \int \frac{1}{h(y)} dy$.

So, $\int \frac{1}{h(y)} dy = \int g(x) dx$, and we avoid our shortcut. (now you see why we often just treat it like a fraction!)

Implicit Solutions

Given $\frac{dy}{dx} = f(x,y)$, with initial condition $y(0) = 2$.

Usually we want to find an explicit solution y of the form: $y(x) =$ (some expression in x).

However, for $x + yy' = 0$ we solve to find an *implicit solution*: $x^2 + y^2 - 4 = 0$.

Then, attempting to solve for y, we end up w/*two solutions*: $y(x) = \pm\sqrt{4 - x^2}$ (so which is it: + or – ??)

Only one of these satisfies the given initial condition: $y(0) = 2$.

So in this case, the solution actually is: $y(x) = \sqrt{4 - x^2}$.

Moral of Story: Not every explicit solution obtained from an implicit solution satisfies all init-conds (so check them!).

Definition: An equation $K(x,y) = 0$ is called an *implicit solution* of a DEQ if it is satisfied by some solution y of the DEQ.

By this definition $(y - 2x)(x^2 + y^2 - 4) = 0$, (where we have multiplied both sides of the above solution by $y - 2x$) is still an implicit solution. However, this new equation also tells us $y = 2x$ is a solution, which *does not solve the original DEQ*!

These erroneous solns can enter our calculations when we use integrating factors while solving a DEQ.
So we must *check the solutions* we obtain with the original DEQ.

We also saw in a previous section that multiplying by an integrating factor can cause us to *lose* singular solutions.
So we must keep track of the different "cases" where we might be dividing by zero.

Examples of Seperable DEQs

Population Growth: $P'(t) = kP$, where $k = \beta - \delta$ and *constants* β, δ are the birth and death rates, respectively.

Radioactive Decay: $N'(t) = -kN$, where N is the number of atoms, and $k > 0$ is the rate at which N decays.

Drug Elimination: $A'(t) = -\lambda A$, where $\lambda > 0$ is the "elimination constant" for that drug.

Underwater Light Intensity: $\frac{dI}{dx} = -kI$, where I is the amount of light, x is the depth below the surface, and $k > 0$ depends upon the properties of the water.

Newton's Law of Cooling (exponential decay)

a cake, cooling

The rate of change (with respect to time) of the Temperature $T(t)$ of an object immersed in an ambient temperature A is proportional to the difference of these temperatures. ...

In other words: $\frac{dT}{dt} = k(A - T)$, where k is a constant that depends upon the heat conductivity of the surrounding medium (air?).
In other words, your cake will cool faster outside in the cold air, than inside in your warm house.

Example: A cake is removed from a 210°F oven, and left to cool in room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F?

$\frac{dT}{dt} = k(A - T) = k(70 - T)$

$\frac{1}{70-T}dT = kdt$, when $T \neq 70$. (what happens when $T = 70$?)

$\int \frac{1}{70-T}dT = \int kdt \qquad -\ln|70 - T| = kt + c$

$kt = -\ln|70 - T| - c$

$e^{kt} = e^{-\ln|70-T|-c} = e^{\ln\left|\frac{1}{70-T}\right|}e^{-c} = \frac{1}{70-T}C$, where $C \neq 0$. Then ??

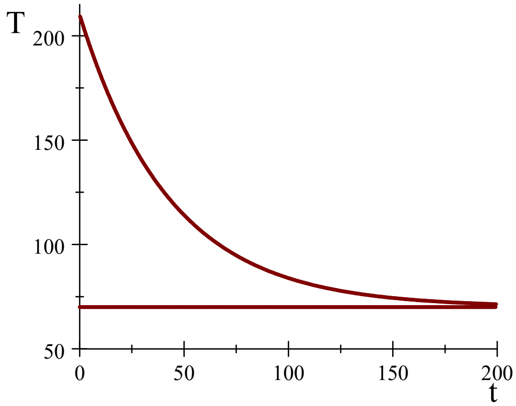
$e^{k\cdot 0} = \frac{1}{70-210}C \quad \Rightarrow \quad -140 = C$. Then ??

$e^{k\cdot 30} = \frac{1}{70-140}(-140) = 2$ (we had two unknown constants, good thing we had two init-conds)

$\Rightarrow \quad 30k = \ln 2, \quad k = \frac{\ln 2}{30}$. Then ??

$$e^{\frac{\ln 2}{30}\cdot t} = \frac{1}{70-100}(-140) = \frac{14}{3}$$

$\frac{\ln 2}{30}t = \ln\frac{14}{3}$, $\quad t = \frac{30\ln\frac{14}{3}}{\ln 2} \approx$ 67 minutes till the cake will be 100°F.



$$e^{\frac{\ln 2}{30}t} = \frac{-140}{70-T}$$

Exercises 1.4 

What did we learn?

- Separable DEQs
- Implicit Solutions