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7.6: Multiple Eigenvalue Solutions: Review

To find the complete general solution to $\vec{x}' = \mathbf{A}\vec{x}$, you need n linearly independent vectors from $\mathbf{A}$'s eigenvalues. But what if one of your eigenvalues λ, has multiplicity k, but only has $k-1$ eigenvectors (so defect $d = 1$) $\{\vec{u}_1, \ldots, \vec{u}_{k-1}\}$. What to do?!?

Algorithm for Eigenvalue of Defect $d = 1$, Multiplicity k

(for example, $\mathbf{A}^{2\times2}$, with $\lambda = 5$, multiplicity 2, but only one eigenvector)

- Compute: $(\mathbf{A} - \lambda\mathbf{I})^2$.
- Choose a nonzero $\vec{v}_2$ so that: $(\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$.
  often in the book $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$, is the zero matrix. Thus, any nonzero vector $\vec{v}_2$ will work,
  so it's traditional to choose a standard vector $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ or $\vec{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.)
- Next, calculate $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_2$ and label this as $\vec{v}_1$.
- If $\vec{v}_1 = \vec{0}$, choose a different nonzero $\vec{v}_2$ above, and recalculate until your $\vec{v}_1$ is nonzero.
- Then, form the k independent solutions:
  $\vec{u}_1 e^{\lambda t}, \ldots, \vec{u}_{k-1} e^{\lambda t}$ and $\left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$. (the last one is the "generalized" eigenvector)

Including the rest of the general solution (for the other eigenvalues): $c_1\vec{w}_1 e^{\lambda_1 t} + \ldots + c_{n-k}\vec{w}_{n-k}e^{\lambda_{n-k}t}$, we end up with:

$$x(t) = c_1\vec{w}_1 e^{\lambda_1 t} + \ldots + c_{n-k}\vec{w}_{n-k}e^{\lambda_{n-k}t} + \left[c_{n-k+1}\vec{u}_1 + \ldots + c_{n-1}\vec{u}_{k-1} + c_n\left(\vec{v}_1 t + \vec{v}_2\right)\right]e^{\lambda t}.$$

Most common case: λ with $k = 2$

- Form the two independent solutions:
  $\vec{u}_1 e^{\lambda t}$ and $\left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$.

So, from the eigenvalue λ, the contribution to the solution is: $c_1\vec{u}_1 e^{\lambda t} + c_2\left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$.

General Algorithm for Eigenvalue of Defect d, Multiplicity k

(for example, $\mathbf{A}^{3\times3}$, with $\lambda = 5$, multiplicity 3, but only one eigenvector, so $d = 2$)

- Compute: $(\mathbf{A} - \lambda\mathbf{I})^{d+1}$.
- Choose a nonzero $\vec{u}_1$ so that: $(\mathbf{A} - \lambda\mathbf{I})^{d+1}\vec{u}_1 = \vec{0}$.
- Successively multiply $\vec{u}_1$ by powers of $(\mathbf{A} - \lambda\mathbf{I})$ until the zero vector is obtained.
- When we first form $(\mathbf{A} - \lambda\mathbf{I})^p\vec{u}_1 = 0$, if the power p is equal to $d+1$, then you now have vectors $\vec{u}_1, \ldots, \vec{u}_{d+1}$ such that:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{u}_1 = \vec{u}_2 \neq \vec{0},$$
$$\vdots$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{u}_d = \vec{u}_{d+1} \neq \vec{0}, \text{ and } (\mathbf{A} - \lambda\mathbf{I})\vec{u}_{d+1} = \vec{0}.$$

  These k vectors are called your generalized eigenvectors.
- If $p < d+1$, then choose a different nonzero $\vec{u}_1$ above until you find $d+1$ nonzero vectors.
- Next, form the $d+1$ independent solutions from:

$$x_1(t) = \vec{u}_{d+1}e^{\lambda t},$$
$$x_2(t) = \left(\vec{u}_{d+1}t + \vec{u}_d\right)e^{\lambda t}.$$
$$\vdots$$

$$x_{d+1}(t) = \left(\vec{u}_{d+1}\frac{t^d}{d!} + \ldots + \vec{u}_3\frac{t^2}{2!} + \vec{u}_2 t + \vec{u}_1\right)e^{\lambda t}.$$

Observe that if our defect has $d+1 < k$, the above algorithm does not produce k linearly independent eigenvectors/solutions. However, if $d+1 < k$, this means that we have additional independent vectors found through the normal process. For example, if we are working with a 4×4 matrix with one eigenvalue λ, and find 2 ordinary vectors $\vec{v}_1, \vec{v}_2$, we have a defect of 2, so $d+1 = 3 < k = 4$. However, we have the 2 ordinary vectors which are linearly independent, in addition to the 3 vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ we find above using our new process. So obviously all 5 of these cannot be linearly independent. In fact, we will also find that $\vec{u}_1$ is a linear combination of $\vec{v}_1, \vec{v}_2$. So, the way we find 4 **linearly independent** vectors here is to find a linear combination of $\vec{v}_1, \vec{v}_2$ (which we will call $\vec{u}_4$) that is linearly independent from $\vec{u}_1$. Then, our basis becomes
$\left\{\vec{u}_1 e^{\lambda t},\ \left(\vec{u}_2 t + \vec{u}_1\right)e^{\lambda t},\ \left(\vec{u}_3 t^2 + \vec{u}_2 t + \vec{u}_1\right)e^{\lambda t},\ \vec{u}_4 e^{\lambda t}\right\}$.
Including the rest of the general solution (for the other eigenvalues): $c_1\vec{w}_1 e^{\lambda_1 t} + \ldots + c_{n-k}\vec{w}_{n-k}e^{\lambda_{n-k}t}$, we end up with: $x(t) = c_1\vec{w}_1 e^{\lambda_1 t} + \ldots + c_{n-k}\vec{w}_{n-k}e^{\lambda_{n-k}t}$

$$+ \left[c_{n-k+1}\vec{u}_{d+1} + c_{n-k+2}\left(\vec{u}_{d+1}t + \vec{u}_d\right) + \ldots + c_{n-k+d-2}\left(\vec{u}_{d+1}\frac{t^d}{d!} + \ldots + \vec{u}_3\frac{t^2}{2!} + \vec{u}_2 t + \vec{u}_1\right)\right]e^{\lambda t}$$
$$+ \left[c_{n-k+d-1}\vec{u}_{d+2} + \ldots + c_n\vec{u}_k\right]e^{\lambda t}.$$

Problem: **#6** Find a general solution to the system: $\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}\vec{x}$.

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) + 16 = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2.$$

$\lambda_1 = \lambda_2 = 5$

$$\begin{bmatrix} 1-5 & -4 \\ 4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y = b \text{ and } x = -b.$$

$\vec{u}_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$

Only one eigenvector? Defective! $d = 1$.

"**Compute**: $(\mathbf{A} - \lambda\mathbf{I})^2$."

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

"**Choose a nonzero vector** $\vec{v}_2$ **so that**: $(\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$."

$(\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$ is therefore satisfied by *any* choice of $\vec{v}_2$.

Generic nonzero choice in these situations: $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

"**Calculate** $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_2$ **and label this** $\vec{v}_1$."

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} =: \vec{v}_1 \qquad \text{(yay, nonzero!)}$$

"**Form the two independent solutions**: $\vec{x}_1(t) = \vec{u}_1 e^{\lambda t}$ **and** $\vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$."

So, $\vec{x}_1(t) = \vec{u}_1 e^{5t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}$.

$$\vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{5t} = \left(\begin{bmatrix} -4 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{5t} = \begin{bmatrix} -4t+1 \\ 4t \end{bmatrix} e^{5t}.$$

Gen. solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t} + c_2\begin{bmatrix} -4t+1 \\ 4t \end{bmatrix} e^{5t}$

$$= \begin{bmatrix} -4c_1 + (-4t+1)c_2 \\ 4c_1 + 4tc_2 \end{bmatrix} e^{5t}.$$

Problem: **#16** **Find a general solution to the system**: $\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix}\vec{x}$.

$|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda)((-2-\lambda)(4-\lambda)+9) = (1-\lambda)(\lambda^2 - 2\lambda + 1)$$

$$= -(\lambda - 1)^3 = 0. \quad \text{OR}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1-\lambda \\ 2 & 3 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & 3 & 1-\lambda \end{vmatrix}$$

$$= (\lambda - 1)^3 = 0.$$

So: $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Now: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

Reductions: $\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{3}{2} & \frac{3}{2} \end{bmatrix}, \qquad z = c,\ y = b,\ x = -\frac{3}{2}b - \frac{3}{2}c$

So, $\left\langle -\frac{3}{2}b - \frac{3}{2}c,\ b,\ c \right\rangle = \langle 3,\ -2,\ 0 \rangle + \langle 3,\ 0, -2 \rangle$, where $b = c = -2$.

$\overrightarrow{u_1} = \langle 3,\ -2,\ 0 \rangle$, and $\overrightarrow{u_2} = \langle 3,\ 0, -2 \rangle$.

Only two eigenvectors? Defective! $\qquad d = 1$.

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

"**Choose a nonzero vector** $\vec{v}_2$ **so that**: $(\mathbf{A} - \lambda\mathbf{I})^2 \vec{v}_2 = \vec{0}$."

Start with generic: $\vec{v}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Calculate $\vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2$.

$$\vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}. \qquad \text{(yay, nonzero!)}$$

Notice that: $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\left((\mathbf{A} - \lambda\mathbf{I})\vec{v}_2\right) = (\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}.$

So, $\vec{v}_1 = \langle 0, -2, 2 \rangle$ is an ordinary eigenvector associated with λ.

Also, recall: $\vec{u_1} = \langle 3, -2, 0 \rangle$ and $\vec{u_2} = \langle 3, 0, -2 \rangle$ are also eigenvectors associated with λ.

So, we might mistakenly think that $\vec{x}(t) = e^t\left[c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{v}_1\right]$ is our general solution.

However, $\vec{v}_1$ is a linear combination of $\vec{u}_1 = \langle 3, -2, 0 \rangle$ and $\vec{u_2} = \langle 3, 0, -2 \rangle$, namely:
$\vec{u}_1 - \vec{u_2} = \vec{v}_1$.

So, $\vec{v}_1 e^t$ is a linear combination of the independent solutions $\vec{u}_1 e^t$ and $\vec{u}_2 e^t$ (and therefore dependent).
So, we must instead use the prescribed $\left(\vec{v}_1 t + \vec{v}_2\right)e^t$ as the desired third **independent** solution.

The corresponding general solution is described by
$\vec{x}(t) = e^t\left[c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\left(\vec{v}_1 t + \vec{v}_2\right)\right]$

OR

$\vec{x_1}(t) = e^t(3c_1 + 3c_2 + c_3)$

$\vec{x_2}(t) = e^t(-2c_1 - 2c_3 t)$

$\vec{x_3}(t) = e^t(-2c_2 + 2c_3 t)$.

Problem - Find a general solution to the system: $\vec{x}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\vec{x}$, where
$\lambda \in \{0, -2, -2, -2\}$, and the eigenvector associated with $\lambda_1 = 0$ is $\vec{u}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$.

$\mathbf{A} - \lambda_2\mathbf{I} = \mathbf{A} + 2\mathbf{I} = 0$

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \overset{R_4+R_3 \text{ and } R_3+R_1}{\Rightarrow} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix},$$

$x_4 = s, \quad x_3 = t, \quad 2x_2 = -s, \quad 2x_1 = -t.$

$$\Rightarrow \quad \vec{x} = \langle -\tfrac{1}{2}t, -\tfrac{1}{2}s, t, s \rangle = \langle 1, 0, -2, 0 \rangle + \langle 0, 1, 0, -2 \rangle, \text{ when } t, s = -2.$$

So: $\vec{u}_2 = \langle 1, 0, -2, 0 \rangle$ and $\vec{u}_3 = \langle 0, 1, 0, -2 \rangle$. Defect $d = 1$.

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

"**Choose a nonzero vector $\vec{v}_2$ so that**: $(\mathbf{A} - \lambda\mathbf{I})^2 \vec{v}_2 = \vec{0}$."

Start with generic: $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$.

$$(\mathbf{A} - \lambda\mathbf{I})^2 \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \neq 0, \text{ so we see this doesn't work.}$$

But looking at $(\mathbf{A} - \lambda\mathbf{I})^2$, and making a more informed choice, we choose $\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}^T$. And indeed:

$$(\mathbf{A} - \lambda\mathbf{I})^2 \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, our next task is to determine nonzero $\vec{v}_1 := (\mathbf{A}-\lambda\mathbf{I})\vec{v}_2$, but observe that

$$(\mathbf{A}-\lambda\mathbf{I})\begin{bmatrix}1 & 0 & -2 & 0\end{bmatrix}^T = \begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ -2 & 2 & -1 & 1\\ 2 & -2 & 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\\-2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$. But we need this to be nonzero, so apparently we must choose a different $\vec{v}_2$ vector above. By looking at $(\mathbf{A}-\lambda\mathbf{I})^2$ again, and making another informed choice, we choose $\vec{v}_2 = \begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix}^T$. And notice that

$$(\mathbf{A}-\lambda\mathbf{I})^2\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix}^T = \begin{bmatrix}2 & 2 & 1 & 1\\ 2 & 2 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$. So that is good. But what about

$\vec{v}_1 := (\mathbf{A}-\lambda\mathbf{I})\vec{v}_2$?

$$\vec{v}_1 := (\mathbf{A}-\lambda\mathbf{I})\vec{v}_2 = \begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ -2 & 2 & -1 & 1\\ 2 & -2 & 1 & -1\end{bmatrix}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\\-2\\2\end{bmatrix}.$$ (yay, nonzero!)

The corresponding general solution is described by

$$\vec{x}(t) = c_1\vec{u}_1e^0 + c_2\vec{u}_2e^{-2t} + c_3\vec{u}_3e^{-2t} + c_4\left(\vec{v}_1t+\vec{v}_2\right)e^{-2t}$$

OR

$$\vec{x}(t) = c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-2\\0\end{bmatrix}e^{-2t} + c_3\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}e^{-2t} + c_4\left(\begin{bmatrix}1\\-1\\-2\\2\end{bmatrix}t + \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right)e^{-2t}$$

Observe that $\vec{v}_1 = \vec{u}_2 - \vec{u}_3$.