MATH 2243: Linear Algebra & Differential Equations

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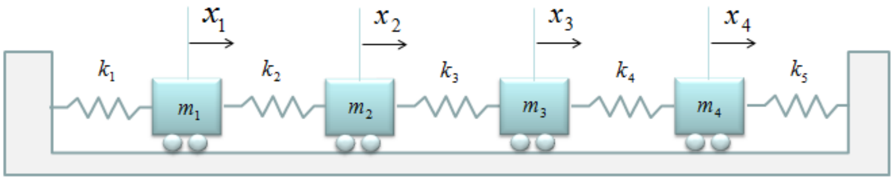
7.5: Second-Order Systems and Mechanical Applications



Masses (n of them) connected to each other and connected to two walls by $n+1$ springs. Assume no friction, and that each mass m_j reacts to the spring(s) attached to it by the familiar formula $F = m_j x_j'' = -kx$. So, assuming the mass in question m_j is reacting to two springs (k_j and k_{j+1}), we have: $F = m_j x_j'' = -k_j(x_j - x_{j-1}) + k_{j+1}(x_{j+1} - x_j)$.

Case: $n = 3$

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1),$$
$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2),$$
$$m_3 x_3'' = -k_3(x_3 - x_2) + k_4 x_3.$$

Observe that the initial k_1 and the final spring k_{n+1} only have one mass displacement effecting it (x_1 and x_n, respectively).

We can put the displacement x_j of each mass m_j into a **displacement vector**: $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.
Similarly with the masses, we have a **mass matrix**:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}.$$

For the spring constants, we have this **stiffness matrix**:

$$\mathbf{K} = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{bmatrix}.$$

Using these mathematical objects, we can more elegantly represent the the above system as $\mathbf{M}\vec{\mathbf{x}}'' = \mathbf{K}\vec{\mathbf{x}}$. Since $\mathbf{M}$ is invertible, we can calculate $\mathbf{M}^{-1}$ and multiply both sides of the equation (on the left) to simplify our equation further to our familiar $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}}$, where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$.

Solution of Second-Order Homogeneous Systems: $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}}$

Consider solutions of the form e^{rt}, which we used for single equations. To solve for a **system**, however, we will need to make this into a vector. Multiplying by a generic constant vector $\vec{v}$, we have $\vec{v}e^{rt}$. Assuming a solution of this form, and plugging it back into our DEQ, we get:
$\mathbf{A}\vec{v}e^{rt} = \left(\vec{v}e^{rt}\right)'' = r\left(\vec{v}e^{rt}\right)' = r^2\vec{v}e^{rt}$.
Dividing by e^{rt}, we get $\mathbf{A}\vec{v} = r^2\vec{v}$. But this is the eigenvector/eigenvalue equation where $\vec{v}$ is an eigenvector for $\mathbf{A}$, and $\lambda = r^2$ is the associated eigenvalue.

Typically, when systems of equations like these model mechanical systems, we have eigenvalues $\lambda_j = -\omega_j^2$ of $\mathbf{A}$ which are less than or equal to zero (where each ω_j is a **circular frequency**). This gives us $r_j = \pm\sqrt{-\omega_j^2} = \pm\omega_j i$. So, for the eigenpair $\lambda_j, \vec{v}_j$ of $\mathbf{A}$ we have: $\vec{v}_j e^{i\omega_j t} = (\cos\omega_j t + i\sin\omega_j t)\vec{v}_j$. And from the real and imaginary parts, we get: $\mathbf{x}_j(t) = (a_j\cos\omega_j t + b_j\sin\omega_j t)\vec{v}_j$.

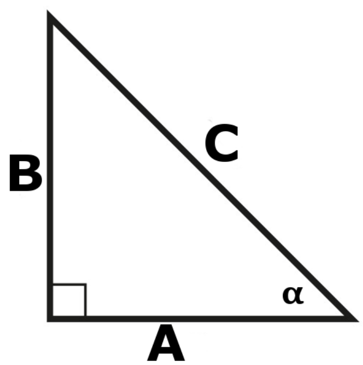
Theorem: If the $n \times n$ matrix $\mathbf{A}$ has n distinct nonpositive eigenvalues $-\omega_1^2,\ -\omega_2^2,\ \ldots,\ -\omega_n^2$, with eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then a general solution of $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}}$ is given by $\vec{\mathbf{x}}(t) = \sum_{j=1}^{n}(a_j\cos\omega_j t + b_j\sin\omega_j t)\vec{v}_j$, where a_j and b_j are arbitrary constants. In the case where $-\omega_j^2 = 0$, the corresponding part $\vec{\mathbf{x}}_j(t)$ of the general solution is $[\ldots + (a_j + b_j t)\vec{v}_j + \ldots]$.

We wish to convert the solution above to the form $\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} c_j\cos(\omega_j t - \alpha_j)\vec{v}_j$, where α_j is the "phase shift" or "phase angle."

So, recall (or learn for the first time) that if we have: $A\cos\omega t + B\sin\omega t$.

and wish to alter it to be like: $C\cos(\omega t - \alpha)$, (where C turns out to be the amplitude of the vibration)

we let A and B be the legs of a right triangle. Then the hypotenus is: $C = \sqrt{A^2 + B^2}$.



With angle α (opposite of B), recall we have: $\cos\alpha = \frac{A}{C}, \quad \sin\alpha = \frac{B}{C}$,

$$\text{where } \alpha = \begin{cases} \tan^{-1}\frac{B}{A} & \text{if } A, B > 0 \text{ (1st quadrant)}, \\ \pi + \tan^{-1}\frac{B}{A} & \text{if } A < 0 \text{ (2nd/3rd quadrant)}, \\ 2\pi + \tan^{-1}\frac{B}{A} & \text{if } A > 0, B < 0 \text{ (4th quadrant)}. \end{cases}$$

Thus we transform into, $A\cos\omega t + B\sin\omega t = C\left(\frac{A}{C}\cos\omega t + \frac{B}{C}\sin\omega t\right)$
$= C(\cos\alpha\cos\omega t + \sin\alpha\sin\omega t)$.

Recall the Trigonometric Identity: $\cos x\cos y + \sin y\sin x = \cos(x - y) = \cos(y - x)$.

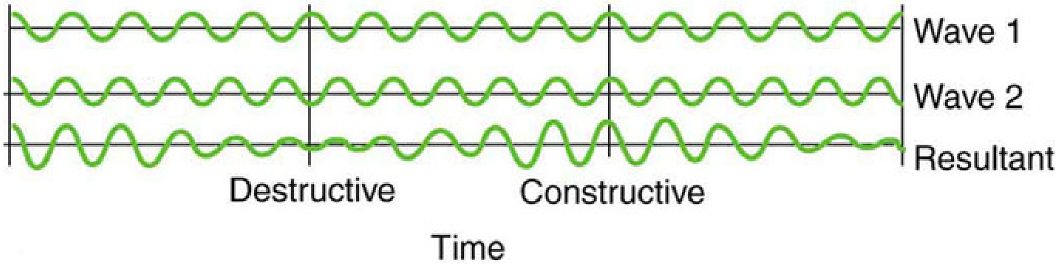
So we get: $C\cos(\omega t - \alpha)$, where C is the **amplitude**,

ω is the **circular frequency** in $\frac{rad}{sec}$, and α is the **phase angle**.

Period of Motion: $T = \frac{2\pi}{\omega}\ sec$. **Frequency**: $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$ in $\frac{cycles}{sec}$.

So returning to $\vec{\mathbf{x}}(t)$, we have $\vec{\mathbf{x}}_j(t) = c_j(\cos\alpha_j\cos 5t + \sin\alpha_j\sin 5t)\vec{v}_j = c_j\cos(5t - \alpha_j)\vec{v}_j$.

Superposition of Wave Frequecies ω_1 and ω_2:



Here is a video showing the kinds of movements involved in this section:
https://www.youtube.com/watch?v=cu4TvUwk17g

Forced Oscillations and Resonance:

Let $\mathbf{M}\vec{\mathbf{x}}'' = \mathbf{K}\vec{\mathbf{x}} + \vec{\mathbf{F}}$ where $\vec{\mathbf{F}} = \left[F_1(t)\ F_2(t)\ \ldots\ F_n(t) \right]^T$ are the external forces acting on the masses $(m_1,\ m_2,\ \ldots,\ m_n)$.

So, $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$, where $\vec{\mathbf{f}} = \left[\frac{F_1(t)}{m_1}\ \frac{F_2(t)}{m_2}\ \ldots\ \frac{F_n(t)}{m_2} \right]^T$ is the external force vector **per unit mass**.

Often the external forces are periodic,

and we have $\vec{\mathbf{f}}(t) = \vec{\mathbf{F}}_0 \cos \omega t$, where $\vec{\mathbf{F}}_0$ is some constant vector.

We obtain **resonance** when the external (forced) frequency ω is equal to one of the system's internal frequencies $\{\omega_1, \omega_2, \ldots, \omega_n\}$.

Undetermined coefficients suggests a trial solution of:

$\vec{\mathbf{x}}_{trial}(t) = \vec{\mathbf{c}} \cos \omega t$. (why not "$+\vec{\mathbf{b}} \sin \omega t$" ??)

We solve for particular solution by plugging in this trial solution, and determining the coefficients: $\vec{\mathbf{c}} = \left[c_1\ c_2\ \ldots\ c_n \right]$.

As with a single equation with forced oscillation, we have a periodic and transient solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_{tr}(t) + \vec{\mathbf{x}}_{sp}(t)$ (see section 5.6). Given any damping, the transient solution eventually disappears leaving only the periodic solution (which is being induced by the external force).

Problem: #7 Suppose a mass-and-spring system have the following **stiffness matrix**…

$$\mathbf{K} = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix}$$

and has the following values for the mass and spring constants...

$$m_1 = m_2 = 1; \qquad k_1 = 4,\ k_2 = 6,\ k_3 = 4.$$

Find the two natural frequencies of the system and describe its two natural modes of oscillation.

$$\mathbf{M}\vec{\mathbf{x}}'' = \mathbf{K}\vec{\mathbf{x}} \qquad \text{or} \qquad \mathbf{x}'' = \mathbf{M}^{-1}\mathbf{K}\vec{\mathbf{x}}.$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \mathbf{M}^{-1}.$$

$$\text{So, } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -(4+6) & 6 \\ 6 & -(6+4) \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}.$$

$$\begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix} = (10-\lambda)^2 - 36 = \lambda^2 + 20\lambda + 64 = (\lambda+16)(\lambda+4).$$

Eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -16$, with associated eigenvectors $v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

Recall: "$\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} (a_j \cos \omega_j t + b_j \sin \omega_j t) \vec{v}_j$" and "Eigenvalues: $\lambda = -\omega_i^2$"

Therefore: $\mathbf{x}(t) = (a_1 \cos 2t + b_1 \sin 2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (a_2 \cos 4t + b_2 \sin 4t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x_1(t) = a_1 \cos 2t + b_1 \sin 2t + a_2 \cos 4t + b_2 \sin 4t,$$
$$x_2(t) = a_1 \cos 2t + b_1 \sin 2t - a_2 \cos 4t - b_2 \sin 4t.$$

"**Describe its two natural modes of oscillation**."

The natural frequencies are $\omega_1 = 2$ and $\omega_2 = 4$. In the natural mode with frequency 2, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequencies 4, they move in opposite directions with equal amplitudes.

Problem: **#10** The mass-and-spring system of the problem #7 (above) is set in motion from rest [$x_1'(0) = x_2'(0) = 0$], at its equilibrium position [$x_1(0) = x_2(0) = 0$], with external forces $F_1(t) = 30 \cos t$ and $F_2(t) = 60 \cos t$ acting on the masses m_1 and m_2, respectively. Find the resulting motion of the system and describe it as a superposition of oscillations.

Recall: $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$, $m_1 = 1$, $m_2 = 1$, and $\mathbf{A} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$.

Observe that $\vec{\mathbf{f}} = \mathbf{M}^{-1}\mathbf{F} = \mathbf{F} = [30 \cos t,\ 60 \cos t\,]$ (since $\mathbf{M} = \mathbf{I}$).

So, forming the nonhomogeneous DEQ $\vec{\mathbf{x}}'' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$, we have:

$$x_1'' = -10x_1 + 6x_2 + 30 \cos t,$$
$$x_2'' = 6x_1 - 10x_2 + 60 \cos t \qquad (*)$$

Recall complementary solution from prob. 7:

$$x_{c,1}(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t$$

$$x_{c,2}(t) = a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t$$

Recall from the review that the "**trial solution is**: $\vec{x}_{trial}(t) = \vec{c} \cos \omega t$," where we can label the components $\vec{c} := \begin{bmatrix} d_1 & d_2 \end{bmatrix}$.

Taking derivatives of the of the trial solution $x_1 = d_1 \cos t, \quad x_2 = d_2 \cos t$ in order to substitute into the system $(*)$:

$x_1' = -d_1 \sin t, \qquad x_2' = -d_2 \sin t, \qquad x_1'' = -d_1 \cos t, \qquad x_2'' = -d_2 \cos t.$

$$(-d_1 \cos t) = -10(d_1 \cos t) + 6(d_2 \cos t) + 30 \cos t,$$

$$(-d_2 \cos t) = 6(d_1 \cos t) - 10(d_2 \cos t) + 60 \cos t.$$

Dividing by $\cos t$:

$$-d_1 = -10 d_1 + 6 d_2 + 30,$$

$$-d_2 = 6 d_1 - 10 d_2 + 60. \qquad \text{(two equations in two unknowns)}$$

$$9d_1 = 6d_2 + 30, \; 9d_2 = 6d_1 + 60; \qquad d_1 = \tfrac{2}{3} d_2 + \tfrac{10}{3}, \; d_2 = \tfrac{2}{3}\left(\tfrac{2}{3} d_2 + \tfrac{10}{3}\right) + \tfrac{20}{3}$$

$$\tfrac{5}{9} d_2 = \tfrac{80}{9}, \; d_2 = 16, \qquad d_1 = \tfrac{2}{3} \cdot 16 + \tfrac{10}{3} = 14.$$

So a general solution is given by:

$$x_1(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t + 14 \cos t,$$

$$x_2(t) = a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t + 16 \cos t. \qquad (**)$$

Initial conditions: $x_1(0) = x_2(0) = 0$

$$0 = a_1 + b_1 + 14, \qquad 0 = a_1 - b_1 + 16;$$

So: $\quad a_1 = -(b_1 + 14), \qquad 0 = -(b_1 + 14) - b_1 + 16,$

$$2b_1 = 2, \; b_1 = 1; \qquad a_1 = -(1 + 14) = -15.$$

Now taking the derivative for the initial condition: $x_1'(0) = x_2'(0) = 0$:

$$x_1' = -a_1 \sin 2t + a_2 \cos 2t - b_1 \sin 4t + b_2 \cos 4t - 14 \sin t,$$

$$x_2' = -a_1 \sin 2t + a_2 \cos 2t + b_1 \sin 4t - b_2 \cos 4t - 16 \sin t.$$

$$0 = a_2 + b_2, \qquad 0 = a_2 - b_2;$$

$$a_2 = b_2, \quad b_2 = -(b_2); \quad b_2 = 0, \quad a_2 = 0.$$

The resulting particular solution from $(**)$ is:

$$x_1(t) = \cos 4t - 15\cos 2t + 14\cos t,$$
$$x_2(t) = -\cos 4t - 15\cos 2t + 16\cos t.$$

"**Describe it as a superposition of oscillations at three different frequencies**."

We have a superposition of three oscillations, in which the two masses:

- Move in opposite directions with frequency $\omega_3 = 4$ and equal amplitudes.
- Move in the same direction with frequency $\omega_2 = 2$ and equal amplitudes;
- Move in the same direction with frequency $\omega_1 = 1$ and with the amplitude of motion of m_2 being 16, and m_1 being 14.

Problem: **#11a** Consider a mass-and-spring system containing two masses $m_1 = 1$ and $m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations: $x'' = -40x + 8y, \quad y'' = 12x - 60y.$
What are the natural frequencies, and in what directions and amplitudes do the masses move?

$$\mathbf{A} = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix},$$

Determining the eigenvalues:

$$\begin{vmatrix} -40-\lambda & 8 \\ 12 & -60-\lambda \end{vmatrix} \Rightarrow (40+\lambda)(60+\lambda) - 96 = \lambda^2 + 100\lambda + 2304$$

$$= (\lambda+64)(\lambda+36). \quad \text{So: } \lambda_{1,2} = -36, -64.$$

$$\lambda_1 = -36 : \begin{bmatrix} -40+36 & 8 \\ 12 & -60+36 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 12 & -24 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 8 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \quad y = s, \text{ and } x = 2s, \text{ so } \vec{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T, \text{ where } s = 1.$$

Similarly for $\lambda_2 = -64$: $\vec{v}_2 = \begin{bmatrix} 1 & -3 \end{bmatrix}^T$.

So we have the general solution: $\vec{x} = (a_1 \cos 6t + b_1 \sin 6t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (a_2 \cos 8t + b_2 \sin 8t)\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

OR

$x(t) = 2a_1 \cos 6t + 2b_1 \sin 6t + a_2 \cos 8t + b_2 \sin 8t,$

$y(t) = a_1 \cos 6t + b_1 \sin 6t - 3a_2 \cos 8t - 3b_2 \sin 8t.$

What are the natural frequencies, and in what directions and amplitudes do the masses move?

Problem: ≈#11b Assume that the two masses above start in motion with the initial conditions: $x(0) = 19,\ x'(0) = 12$, and $y(0) = 3,\ y'(0) = 6$, with no external force. Describe the resulting motion as a superposition of oscillations at two different frequencies.

Applying the first set of initial conditions:

$20 = 2a_1 \cos 0 + 2b_1 \sin 0 + a_2 \cos 0 + b_2 \sin 0,$

$3 = a_1 \cos 0 + b_1 \sin 0 - 3a_2 \cos 0 - 3b_2 \sin 0.$

Simplifying:

$20 = 2a_1 + a_2, \qquad 3 = a_1 - 3a_2.$

Solving two equations in two unknowns:

$a_1 = 3 + 3a_2, \qquad 20 = 2(3 + 3a_2) + a_2 = 6 + 7a_2, \qquad a_2 = 2$

$a_1 = 3 + 6 = 9$

$x'(t) = -12a_1 \sin 6t + 12b_1 \cos 6t - 8a_2 \sin 8t + 8b_2 \cos 8t,$

$y'(t) = -6a_1 \sin 6t + 6b_1 \cos 6t + 24a_2 \sin 8t - 24b_2 \cos 8t.$

Applying the derivative initial conditions:

$12 = -12a_1 \sin 0 + 12b_1 \cos 0 - 8a_2 \sin 0 + 8b_2 \cos 0,$

$6 = -6a_1 \sin 0 + 6b_1 \cos 0 + 24a_2 \sin 0 - 24b_2 \cos 0.$

Simplifying:

$12 = 12b_1 + 8b_2,$

$6 = 6b_1 - 24b_2.$

Solving two equations in two unknowns:

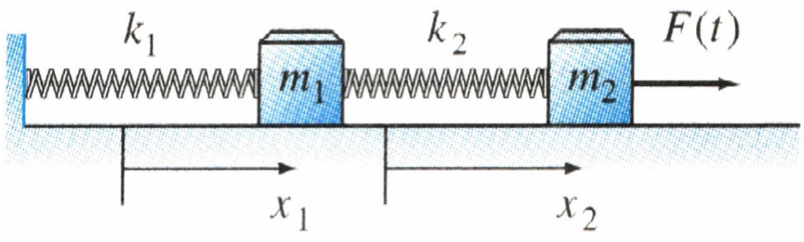
$b_1 = 1 - 4b_2, \qquad 12 = 12(1 - 4b_2) + 8b_2 = 12 - 40b_2, \qquad b_2 = 0,$

$b_1 = 1.$

So: $x(t) = 18\cos 6t + 2\sin 6t + 2\cos 8t,$

$y(t) = 9\cos 6t + \sin 6t - 6\cos 8t.$

Describe the resulting motion as a superposition of oscillations at three different frequencies.



Problem: #15. **Suppose that** $m_1 = 2,\ m_2 = \frac{1}{2},\ k_1 = 75,\ k_2 = 25,\ \vec{\mathbf{F}}_0 = \begin{bmatrix} 0 & 100 \end{bmatrix}$, **and** $\omega = 10$ **(all in** *mks* **units) in the forced mass-and-spring system shown. Find the solution of the system** $\mathbf{M}\vec{x}'' = \mathbf{K}\vec{x} + \mathbf{F}$ **that satisfies the initial conditions** $\vec{x}(0) = \vec{x}'(0) = \vec{0}$.

Recall: For the spring constants, we have this stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix} = \begin{bmatrix} -100 & 25 \\ 25 & -25 \end{bmatrix}.$$

Mass matrix: $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$

First we need the general solution of the homogeneous system $\vec{x}'' = \mathbf{M}^{-1}\mathbf{K}\vec{x}$, with

$$\mathbf{M}^{-1}\mathbf{K} = \mathbf{A} = \begin{bmatrix} -50 & \frac{25}{2} \\ 50 & -50 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -25$ and $\lambda_2 = -75$, so the natural frequencies of the system are $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$. Associated eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$.

So the complementary solution $\vec{x}_c(t)$ is given by…

$$x_1(t) = (a_1\cos 5t + b_1\sin 5t) + \left(a_2\cos 5\sqrt{3}\,t + b_2\sin 5\sqrt{3}\,t\right),$$

$$x_2(t) = (2a_1\cos 5t + 2b_1\sin 5t) - \left(2a_2\cos 5\sqrt{3}\,t + 2b_2\sin 5\sqrt{3}\,t\right).$$

Trial solution to "$\vec{\mathbf{F}}_0 = \begin{bmatrix} 0 & 100 \end{bmatrix}^T$, and $\omega = 10$" is...

Recall that: $\vec{x}'' = \mathbf{A}\vec{x} + \vec{\mathbf{f}} = \mathbf{M}^{-1}\mathbf{K}\vec{x} + \mathbf{M}^{-1}\vec{\mathbf{F}}_0 \cos\omega t = \mathbf{M}^{-1}\mathbf{K}\vec{x} + \begin{bmatrix} 0 & 200 \end{bmatrix}^T \cos 10t$

(note from image above that $\vec{\mathbf{F}}_0$ is only directly affecting $m_2\ldots$).

So trial solution: $\vec{x}_{trial}(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T \cos 10t$, and we find...

$$\vec{x}'_{trial} = -10\vec{c}\sin 10t, \qquad \vec{x}''_{trial} = -100\vec{c}\cos 10t.$$

$$\vec{x}''_{trial} = \mathbf{A}\vec{x}_{trial} + \begin{bmatrix} 0 & 200 \end{bmatrix}^T \cos 10t$$

Substituting...

$$-100\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\cos 10t = \begin{bmatrix} -50 & \frac{25}{2} \\ 50 & -50 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\cos 10t + \begin{bmatrix} 0 \\ 200 \end{bmatrix}\cos 10t,$$

$$\Rightarrow \quad \begin{bmatrix} -100c_1 \\ -100c_2 \end{bmatrix} = \begin{bmatrix} -50c_1 + \frac{25}{2}c_2 \\ 50c_1 - 50c_2 + 200 \end{bmatrix}, \qquad \text{(two equations in two unknowns)}$$

$-50c_1 = \frac{25}{2}c_2, \qquad c_1 = -\frac{1}{4}c_2$

$-50c_2 = 50c_1 + 200 = 50\left(-\frac{1}{4}c_2\right) + 200$

$c_2 = \frac{1}{4}c_2 - 4, \qquad \frac{3}{4}c_2 = -4, \qquad c_2 = -\frac{16}{3}$ and $c_1 = \frac{4}{3}$.

So a particular solution $\vec{x}_{sp}(t)$ is described by...

$$x_{sp_1}(t) = \frac{4}{3}\cos 10t, \qquad x_{sp_2}(t) = -\frac{16}{3}\cos 10t.$$

General Solution:

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_{sp}(t)$$

$$x_1(t) = (a_1\cos 5t + a_2\sin 5t) + \left(b_1\cos 5\sqrt{3}\,t + b_2\sin 5\sqrt{3}\,t\right) + \frac{4}{3}\cos 10t,$$

$$x_2(t) = (2a_1\cos 5t + 2a_2\sin 5t) - \left(2b_1\cos 5\sqrt{3}\,t + 2b_2\sin 5\sqrt{3}\,t\right) - \frac{8}{3}\cos 10t.$$

" **Initial conditions** $\vec{x}(0) = \vec{x}'(0) = \vec{0}$ "

Finally, when we impose the initial conditions on the solution $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_{sp}(t)$

$$0 = (a_1\cos(0) + 0) + (b_1\cos(0) + 0) + \frac{2}{3}\cos(0) = a_1 + b_1 + \frac{4}{3},$$

$$0 = (2a_1 \cos(0) + 0) - (2b_1 \cos(0) + 0) - \tfrac{8}{3}\cos(0) = 2a_1 - 2b_1 - \tfrac{16}{3}.$$

$$a_1 = -b_1 - \tfrac{4}{3}, \qquad 2b_1 = 2(-b_1 - \tfrac{4}{3}) - \tfrac{16}{3}, \qquad 4b_1 = -8,$$
$$b_1 = -2, \qquad a_1 = \tfrac{2}{3}.$$

We find that $a_1 = \tfrac{2}{3}$, $a_2 = 0$, $b_1 = -2$, and $b_2 = 0$.

Thus the solution we seek is described by…

$$x_1(t) = \tfrac{2}{3}\cos 5t - 2\cos 5\sqrt{3}\,t + \tfrac{4}{3}\cos 10t,$$
$$x_2(t) = \tfrac{4}{3}\cos 5t + 4\cos 5\sqrt{3}\,t - \tfrac{16}{3}\cos 10t.$$

We have a superposition of 2 natural oscillations with the frequencies $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$ and forced oscillation with $\omega = 10$. In each of the two natural oscillations the amplitude of motion of m_2 is twice that of m_1, while in the forced oscillation the amplitude of motion of m_2 is four times that of m_1. Regarding direction of motion, in oscillation $\omega = 5$ the masses are moving in the same direction, while in the other two oscillations they are moving in opposite directions.