MATH 2243: **Linear Algebra** & **Differential Equations**

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7.1: First-Order Systems and Applications

We have discussed ways of solving DEQs involving one dependent variable: $y' = f(y,t)$, $x'' = f(x',x,t)$, etc.

In this chapter, we will start to learn how to solve systems of DEQs involving several dependent variables. In particular, systems which have as many equations as dependent variables. For example, this model of two (x,y) vibrating atoms:

$$\begin{aligned} m_1x'' &= -k_1x + k_2(y-x) \\ m_2y'' &= -k_2(y-x) + f(t) \end{aligned}$$

The matrix manipulation skills we learned will help us to solve these types of problems in coming sections.

To simplify these problems, however, we first introduce a technique to reduce the order of the DEQ. In particular, we have more tools to solve first-order than higher-order DEQs. And it turns out we can transform higher-order into first-order DEQs.

Transforming Higher-Order DEQs into a System of First-Order DEQs

If you're given: $x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$

Then, define some new variables: $x_0 := x, \qquad x_1 := x' \;\; (= x_0'), \qquad x_2 := x'' \;\; (= x_1'),$

(note below that you don't need the highest derivative $x^{(3)}$ to be represented by a new variable)

Using this "dictionary," we can define a first-order system to replace our given DEQ.

Particularly, the first two DEQs listed below simply come from the dictionary itself.

The last DEQ is the given DEQ, but with the variable swapped out for the new ones.

$$x_0' = x_1,$$

$$x_1' = x_2,$$

$$x_2' + 3x_2 + 2x_1 - 5x_0 = \sin 2t.$$

(notice that the old variable x is eliminated completely)

Why would we make this transformation? Because first-order DEQs are easier to solve, especially for systems of DEQs. The price we pay is that now we have to solve three of them! Which is part of why we learned about matrices.

First-Order System Transformed to Second-Order DEQ

Now let's go the other way around, turning a system of first-order DEQs into one higher-order DEQ.

Why would we make this transformation? Notice the system below is coupled (e.g., there is a y in the x' DEQ). For simple low dimensional coupled systems like this one, instead of using matrix methods (eigenvalues/eigenvectors), it may be simpler to do the transformation below and solve the resulting DEQ using the simpler methods learned previously in the course.

Transforming and solving a 2D System: Example, $x' = -2y, \qquad y' = \frac{1}{2}x.$

- ♦ Attempt to alter the equations toward the aim of substituting one into the other.

- ♦ The resulting equation should be either in x or y, depending on your choice above.

  For example, above you can differentiate the first DEQ,

  and then substitute the second DEQ into it:

  $$x'' = -2y' = -2\left(\tfrac{1}{2}x\right) = -x, \qquad \Rightarrow \qquad x'' + x = 0.$$

- ♦ Next, use your standard techniques to solve for $x(t)$.

- ♦ Finally, use substitution again to plug $x(t)$ into $x' = -2y$ and solve for $y(t)$.

Existence and Uniqueness of Solutions for Linear Systems

Given: $x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + f_1(t),$

$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + f_2(t),$

$x_3' = p_{31}(t)x_1 + p_{32}(t)x_3 + f_3(t),$

with initial conditions: $x_1(a) = b_1, \quad x_2(a) = b_2, \quad x_3(a) = b_3.$

If $p_{11},\ p_{12},\ p_{21},\ p_{22},\ p_{31},\ p_{32},\ f_1,\ f_2,\ f_3$ are continuous on some I containing $t = a$,

then there **exists** a **unique** solution to the system $(x_1(t), x_2(t), x_3(t))$ within the interval I.

Examples of Two-Dimensional Systems

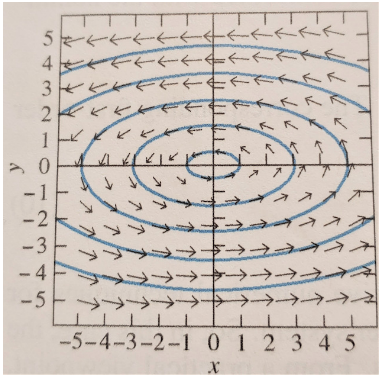
$$x' = -2y,$$
$$y' = \tfrac{1}{2}x.$$

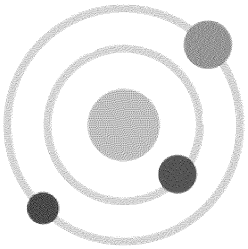
Taking the derivative of the first EQ: $x'' = -2y' = -2\left(\tfrac{1}{2}x\right) = -x.$

So: $x'' + x = 0 \quad \Rightarrow \quad r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad e^{it} = \cos t + i \sin t$

$\Rightarrow \quad x(t) = A\cos t + B\sin t.$

From our given equations: $y = -\tfrac{1}{2}x' = \tfrac{1}{2}(A\sin t - B\cos t).$



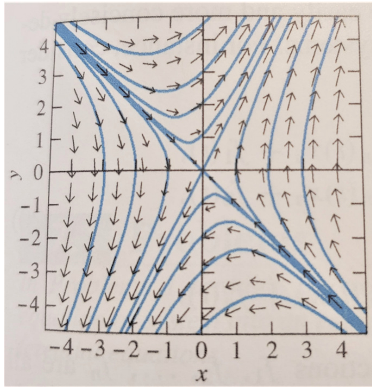


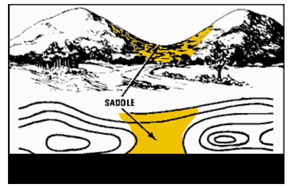
planetary orbits, anything cyclic

And similarly for

$$x' = y,$$
$$y' = 2x + y,$$

we find solution curves like:



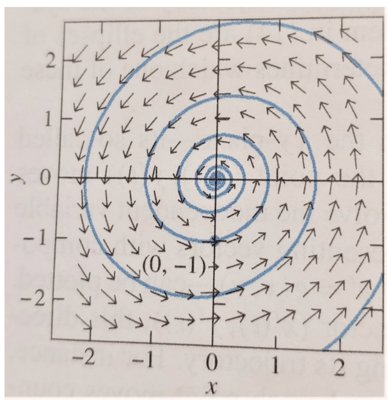


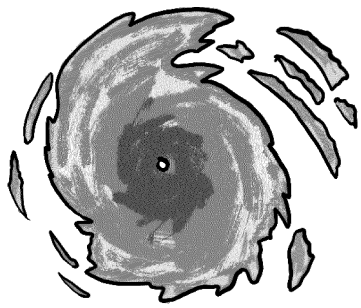
topography

And for

$$x' = -y,$$
$$y' = \frac{101}{100}x - \frac{1}{5}y,$$

we find solution curves like:





Hurricanes, fluidic motion, decaying orbit

Why do such similar systems generate such different solution curves?

Stay tuned for an answer in subsequent sections.

Exercises 

Problem: **#8** Transform the following system into an equivalent system of first-orderDEQs.

$$x'' + 3x' + 4x - 2y = 0, \qquad y'' + 2y' - 3x + y = \cos t.$$

Dictionary: $x_0 := x$, $x_1 := x' = x_0'$, and $y_0 := y$, $y_1 := y' = y_0'$.

(note that you don't need the highest derivatives x'' or y'')

Therefore,

$x_0' = x_1$, $\quad y_0' = y_1$, $\quad$ (from the dictionary)

$x_1' = -4x_0 - 3x_1 + 2y_0$, $\quad y_1' = 3x_0 - 2y_1 - y_0 + \cos t$. $\quad$ (from the given DEQs)

I wrote the new DEQs this way anticipating the following form:

Let $\vec{x} := (x_0, x_1, y_0, y_1)$. Then $\vec{x}' = \mathbf{A}\vec{x} + \vec{v}$ OR

$$\begin{bmatrix} x_0' \\ x_1' \\ y_0' \\ y_1' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{bmatrix}.$$

Problem: **#19** Find the general solution of the following system. Then, find the corresponding particular solution.

$$x' = -y, \qquad y' = 13x + 4y, \qquad x(0) = 0, \qquad y(0) = 3.$$

Note that they are coupled. Differentiating the first equation gives us:

$x'' = -y' = -(13x + 4y) = \ldots$

$= -13x + 4x'$. (we eliminated y !)

So, $x'' - 4x' + 13x = 0$

$r^2 - 4r + 13 = 0$

$r = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$

$e^{(2+3i)t} = e^{2t}(\cos 3t + i \sin 3t)$

$x(t) = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$ Then...?

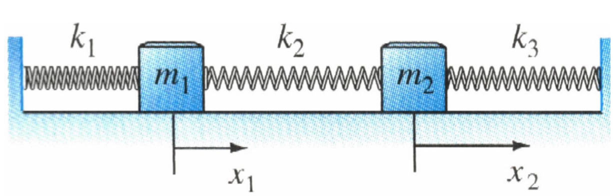
$x(0) = 0$ gives $0 = c_1 e^{2t}$, so $c_1 = 0$. Therefore, $x(t) = c_2 e^{2t} \sin 3t$.

$y = -x'$ gives $y(t) = -(2c_2 e^{2t} \sin 3t + 3c_2 e^{2t} \cos 3t)$. And...?

$y(0) = 3$, so: $3 = -3c_2$, and $c_2 = -1$.

Therefore, $x_p(t) = -e^{2t} \sin 3t$ and $y_p(t) = e^{2t}(2 \sin 3t + 3 \cos 3t)$

Problem: **#24** Derive the DEQs: $m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$ and $m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$ for the displacements (from equilibrium) of the two masses shown below:



Recall Newton's second law (for spring constant k): $\quad mx'' = -kx.$

Looking at the above figure, we see that the first mass is pulled to the left (negatively) by the first spring and to the right (positively) by the second spring. The second mass is pulled to the left (negatively) by the second spring and to the right (positively) by the third spring. We also have to take into account the degree each spring is being stretched (to the right). For instance, the second spring is being stretched to the right a distance of x_2 by mass m_2, but the stretch is lessened by the distance x_1 traveled by m_1. So, the second spring is stretched: $x_2 - x_1$.

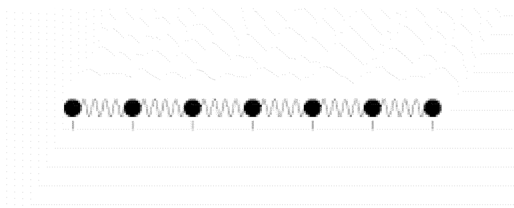
Hence for the first mass Newton's second law gives:

$$m_1 x_1'' = -k_1(x_1) + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2,$$

and for the second mass:

$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(0 - x_2) = k_2 x_1 - (k_2 + k_3)x_2.$$

What can you do with this?



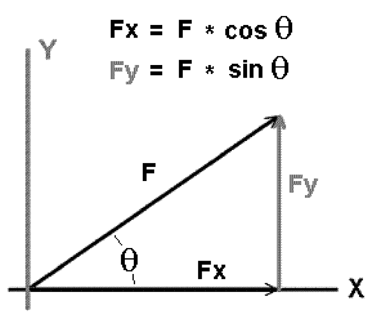
atoms vibrating animation

Problem: #27 A particle of mass m moves in the plane with coordinates $\big(x(t),\ y(t)\big)$ under the influence of a force that is directed toward the origin and has magnitude $\frac{k}{x^2+y^2}$, an inverse-square central force field. Show that:

$$mx'' = -\frac{kx}{r^3} \text{ and } my'' = \frac{ky}{r^3}, \text{ where } r = \sqrt{x^2 + y^2}.$$

Let θ be the polar angular coordinate of the point (x, y) and write $F = \frac{k}{x^2+y^2} = \frac{k}{r^2}$.

Also, let F_x, F_y be the components of force in the x, y directions, respectively.



Recall that $\cos\theta$ and $\sin\theta$ give the horizontal and vertical components of the vector lying on the unit circle pointing toward (x, y). Therefore, they can be represented as $\frac{x}{r}, \frac{y}{r}$, where $r = \sqrt{x^2 + y^2}$.

Then Newton's second law gives:

$$mx'' = -F\cos\theta = -\frac{k}{r^2} \bullet \frac{x}{r} = -\frac{kx}{r^3},$$

$$my'' = -F\sin\theta - \frac{k}{r^2} \bullet \frac{y}{r} = -\frac{ky}{r^3}.$$