**MATH 2243**: **Linear Algebra** & **Differential Equations**

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# 6.3 - Applications Involving Powers of Matrices

If $\mathbf{A} = \mathbf{PDP}^{-1}$, then $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$.

Continuing in this fashion, we find: $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$ with $\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$.

**Transition Matrices**:

A matrix $\mathbf{A}$ which "transitions" the state of a system (notated as a vector $\vec{x}_k$) from one moment $\vec{x}_k$, to the next $\vec{x}_{k+1}$. So, $\vec{x}_{k+1} = \mathbf{A}\vec{x}_k$ where $\vec{x}_0$ is the **initial vector**.
Therefore, $\vec{x}_k = \mathbf{A}^k\vec{x}_0$. Because, $\vec{x}_k = \mathbf{A}\vec{x}_{k-1} = \mathbf{A}^2\vec{x}_{k-2} = \ldots = \mathbf{A}^k\vec{x}_0$.

**Predator-Prey Models**: $F :=$ Foxes, $R :=$ Rabbits.

$F_{k+1} = aF_k + bR_k$
$R_{k+1} = -rF_k + cR_k$ where $a, b, c, r \in \mathbb{R}$.
Also $r$ is rate at which the rabbits are eaten by foxes.

Given this scenario, set up vector and matrix:

$\vec{x}_k = \begin{bmatrix} F_k \\ R_k \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a & b \\ -r & c \end{bmatrix}$.

As $k \to \infty\ldots$

- $F_k$ and $R_k$ may approach constant nonzero values,
- $F_k$ and $R_k$ may both approach zero,
- $F_k$ and $R_k$ may both increase without bound.

**Cayley-Hamilton Theorem**:

"Every square matrix satisfies its own characteristic equation."
If $\mathbf{A}$ has the characteristic polynomial: $p(\lambda) = (-1)^3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$ , then …

$p(\mathbf{A}) = -\mathbf{A}^3 + c_2\mathbf{A}^2 + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}$.

**Problem**: **#28** A city/suburban population transition matrix $\mathbf{A}$ is given.

Find the resulting long-term distribution of a constant total population between the city and its suburbs.

$$C_{k+1} = 0.8C_k + 0.1S_k$$
$$S_{k+1} = 0.2C_k + 0.9S_k$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

**Characteristic polynomial**: $p(\lambda) = \lambda^2 - \frac{17}{10}\lambda + \frac{7}{10} = \frac{1}{10}(\lambda - 1)(10\lambda - 7)$.

**Eigenvalues**: $\lambda_1 = 1,\ \lambda_2 = \frac{7}{10}$

With $\lambda_1 = 1$ : $\left.\begin{array}{r} -\frac{1}{5}a + \frac{1}{10}b = 0 \\ \frac{1}{5}a - \frac{1}{10}b = 0 \end{array}\right\}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

With $\lambda_2 = \frac{7}{10}$ : $\left.\begin{array}{r} -\frac{1}{10}a + \frac{1}{10}b = 0 \\ \frac{1}{5}a + \frac{1}{5}b = 0 \end{array}\right\}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

# Recall:

**Transition Matrices**:

$\vec{x}_{k+1} = \mathbf{A}\vec{x}_k$ where $\vec{x}_0$ is the **initial vector**.

Therefore, $\vec{x}_k = \mathbf{A}^k\vec{x}_0$.

$$\vec{x}_k = \mathbf{A}^k \bullet \vec{x}_0 = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\vec{x}_0$$

$$\Rightarrow \quad \vec{x}_\infty = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\vec{x}_0$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} C_0 + S_0 \\ 2C_0 + 2S_0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} P_0 \\ 2P_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}P_0 \\ \frac{2}{3}P_0 \end{bmatrix}, \text{ as } k \to \infty \text{ where } P_0 \text{ is the total initial population.}$$

Thus the long-term distribution of population is $\frac{1}{3}$ city, $\frac{2}{3}$ suburban.

**Problem**: #32 This problem deals with a fox-rabbit population.

Initially, there are $F_0$ foxes and $R_0$ rabbits; after $k$ months, there are $F_k$ foxes and $R_k$ rabbits.

It involves the transition matrix $\mathbf{A} = \begin{bmatrix} 0.6 & 0.5 \\ -r & 1.2 \end{bmatrix}$ where the kill rate $r$ is $0.175$.

Show that in the long term, the populations of foxes and rabbits both die out.

**Characteristic polynomial**: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{323}{400} = \frac{1}{400}(20\lambda - 19)(20\lambda - 17)$.

**Eigenvalues**: $\lambda_1 = \frac{19}{20}, \ \lambda_2 = \frac{17}{20}$.

With $\lambda_1 = \frac{19}{20}$ :
$$\left.\begin{aligned} -\tfrac{7}{20}a + \tfrac{1}{2}b &= 0 \\ -\tfrac{7}{40}a + \tfrac{1}{4}b &= 0 \end{aligned}\right\} \quad \vec{v}_1 = \begin{bmatrix} 10 \\ 7 \end{bmatrix}.$$

With $\lambda_2 = \frac{17}{20}$ :
$$\left.\begin{aligned} -\tfrac{1}{4}a + \tfrac{1}{2}b &= 0 \\ -\tfrac{7}{40}a + \tfrac{7}{20}b &= 0 \end{aligned}\right\} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\mathbf{P} = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{19}{20} & 0 \\ 0 & \frac{17}{20} \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}$$

$$\vec{x}_k = \mathbf{A}^k\vec{x}_0 = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} \frac{19}{20} & 0 \\ 0 & \frac{17}{20} \end{bmatrix}^k \cdot \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\vec{x}_0$$

$$\rightarrow \vec{x}_\infty = \frac{1}{4}\begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\vec{x}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as $k \rightarrow \infty$. Thus the Fox and Rabbit population both die out.

**Problem**: #37 Suppose that $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Show that $\mathbf{A}^{4n} = \mathbf{I}$, $\mathbf{A}^{4n+2} = -\mathbf{I}$, and $\mathbf{A}^{4n+3} = -\mathbf{A}$, for every positive integer $n$.

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}.$$

So $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = -\mathbf{I}\mathbf{A} = -\mathbf{A}$,

$\mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = (-\mathbf{A})\mathbf{A} = -\mathbf{A}^2 = -(-\mathbf{I}) = \mathbf{I}$, and so forth.

Therefore, $\mathbf{A}^{4n} = \mathbf{I}$, $\mathbf{A}^{4n+2} = -\mathbf{I}$, and $\mathbf{A}^{4n+3} = -\mathbf{A}$, for every positive integer $n$.