MATH 2243: Linear Algebra & Differential Equations

Instructor: Jodin Morey moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

6.2: Diagonals of Matrices

Motivation

$$\vec{x} = \begin{bmatrix} 2,000 \\ 1,500 \\ 10,000 \end{bmatrix}$$



In many applications (like population models), it is possible to discover a **transition matrix A**, that will transition a vector $\vec{x}$ (for example, one containing the populations of different animals in a region) from one state $\vec{x}_0$ to another state $\vec{x}_1$ (for example, from the populations in some year, to the populations in the next year). It is also used in onboard aviation software, satellite orbit maintenance, various statistics applications, and many other fields.

This is done by simply multiplying $\mathbf{A}\vec{x}_0 = \vec{x}_1$. However, we are usually interested in the long-term behavior of $\vec{x}$ (the population), so perhaps what $\vec{x}_{1000}$ will be.

But this would require us to calculate $\overset{\text{(1000 times)}}{\mathbf{AA}\ldots\mathbf{A}}\ \vec{x}_0 = \mathbf{A}^{1000}\vec{x}_0 = \vec{x}_{1000}$.

But $\mathbf{A}^{1000}$ is very difficult to calculate without some better technique. Diagonalization is that technique.

Diagonal Matrix: $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$,

"Zeros off of the diagonal."

So, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal!!

If we can characterize $\mathbf{A}$ as $\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$, where $\mathbf{D}$ is a diagonal matrix,

and $\mathbf{P}, \mathbf{P}^{-1}$ are invertible matrices, then notice that:

$$\mathbf{A}^3 = (\mathbf{P}^{-1}\mathbf{D}\mathbf{P})(\mathbf{P}^{-1}\mathbf{D}\mathbf{P})(\mathbf{P}^{-1}\mathbf{D}\mathbf{P}) = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}\mathbf{P}^{-1})\mathbf{D}(\mathbf{P}\mathbf{P}^{-1})\mathbf{D}\mathbf{P}$$

$$= \mathbf{P}^{-1}\mathbf{D}(\mathbf{I})\mathbf{D}(\mathbf{I})\mathbf{D}\mathbf{P} = \mathbf{P}^{-1}\mathbf{D}\mathbf{D}\mathbf{D}\mathbf{P} = \mathbf{P}^{-1}\mathbf{D}^3\mathbf{P}.$$

Also note that for any diagonal matrix: $\begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \vdots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix}^n = \begin{bmatrix} a_1^n & 0 & \ldots & 0 \\ 0 & a_2^n & \vdots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n^n \end{bmatrix}$.

For example: $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^3 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 25 \end{bmatrix}$

$$= \begin{bmatrix} 27 & 0 \\ 0 & 125 \end{bmatrix}.$$

As a result, to calculate $\mathbf{A}^{1000}$, we need only calculate $\mathbf{D}^{1000}$,
by raising each component to the 1000*th* power, and then
compute $\mathbf{P}^{-1}\mathbf{D}^{1000}\mathbf{P}$ (two matrix multiplications instead of 1000).

But how can we transform $\mathbf{A}$ into $\mathbf{P}^{-1}\mathbf{DP}$? This transformation is called **diagonalizing**.

Diagonalizing Criteria

The $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are called **similar**, if there exists an invertible matrix $\mathbf{P}$, such that: $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.

Diagonalizable: An $n \times n$ matrix $\mathbf{A}$ is called diagonalizable if it is similar to a diagonal matrix $\mathbf{D}$; that is, there exists a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$, and so $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

Criteria for Diagonalizability: An $n \times n$ matrix $\mathbf{A}$ is **diagonalizable** if and only if ($\Leftrightarrow$) it has n linearly independent eigenvectors $\vec{v}_i$ (note that this may be possible even if you have less than n distinct eigenvalues λ_k).

Proof: Due to the "if and only if" ($\Leftrightarrow$), there will be two parts to our proof.
First we will show ($\Leftarrow$), that if we have n linearly independent eigenvectors, then $\mathbf{A}$ as diagonalizable.

That is, we must show the existence of $\mathbf{D}, \mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Suppose our eigenvalues are $\lambda_1, \ldots, \lambda_n$ (perhaps not all unique) corresponding to the

n independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, and let $\mathbf{P} := \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$.

Then, $\mathbf{AP} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_n \\ | & | & & | \end{bmatrix}$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}.$$ (by definition of eigenvector)

Now consider the diagonal matrix $\mathbf{D} := \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$.

So, $\mathbf{PD} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \dots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$.

And note that above we have shown that $\mathbf{AP} = \mathbf{PD}$.

And since we know that $\mathbf{P}$ is invertible (having n linearly independent column vectors), we can multiply on the right by $\mathbf{P}^{-1}$, to obtain: $\mathbf{A} = \mathbf{PDP}^{-1}$. So we have shown ($\Leftarrow$).

Next we will show ($\Rightarrow$) that if $\mathbf{A}$ as diagonalizable, then we have n linearly independent eigenvectors.

Suppose that $\mathbf{A}$ is similar to $\mathbf{D} := \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$, and let $\mathbf{P} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$

be an invertible matrix such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ or equivalently $\mathbf{AP} = \mathbf{PD}$ (this is what it means to be diagonalizable).

We must show that $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors and linearly independent.

We calculate, $\mathbf{AP} = \mathbf{A}\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}\vec{v}_1 & \dots & \mathbf{A}\vec{v}_n \end{bmatrix}$,

and $\mathbf{PD} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = \begin{bmatrix} d_1\vec{v}_1 & \dots & d_n\vec{v}_n \end{bmatrix}$.

Comparing $\mathbf{AP} = \mathbf{PD}$ component-wise, it follows that $\mathbf{A}\vec{v}_j = d_j\vec{v}_j$ for $j = 1, 2, \dots, n$.

But this defines what it is to be in eigenvector/eigenvalue. Thus the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are eigenvectors of $\mathbf{A}$ associated with the eigenvalue $d_1, d_2, \dots, d_n$, respectively.

And it follows from previous theorems that these n eigenvectors of the matrix $\mathbf{A}$ are linearly independent, because they are the column vectors of the invertible matrix **P**. ■

So we have proven the claim that: "An $n \times n$ matrix **A** is **diagonalizable** if and only if (⇔) it has n linearly independent eigenvectors $\vec{v}_i$."

So our ability to diagonalize depends upon **A** having n linearly independent eigenvectors.

The following theorem is helpful in this regard:

Eigenvectors Associated with Distinct Eigenvalues Theorem: Suppose that the eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ are associated with the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of the matrix **A**. Then these k eigenvectors are linearly independent.

Proof: Using induction on k. Obviously true when $k = 1$, so this satisfies our base case.

Our next task is to show that if any set of $k - 1$ eigenvectors associated with distinct eigenvalues is

linearly independent, then any set of k eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{v}_k\}$ is also linearly independent.

In other words, that $c_1\vec{v}_1 + c_2\vec{v}_2, + \ldots + c_k\vec{v}_k = \vec{0}$, $(*)$

requires $c_1 = c_2 = \ldots = c_k = 0$. First note that $(\mathbf{A} - \lambda_j\mathbf{I})\vec{v}_j = 0$, for all j since these are eigenvectors.

For our next calculation we observe that $(\mathbf{A} - \lambda_j\mathbf{I})\vec{v}_i = \mathbf{A}\vec{v}_i - \lambda_j\vec{v}_i = \lambda_i\vec{v}_i - \lambda_j\vec{v}_i = (\lambda_i - \lambda_j)\vec{v}_i$, for all i,j.

Therefore, if we multiply $(*)$ by $(\mathbf{A} - \lambda_1\mathbf{I})$, we get $(\lambda_2 - \lambda_1)c_2\vec{v}_2, + \ldots + (\lambda_k - \lambda_1)c_k\vec{v}_k = \vec{0}$.

Since we assumed the eigenvalues were distinct, and since we assumed that any set of $k - 1$ eigenvectors is

linearly independent, this requires $c_2 = c_3 = \ldots = c_k = 0$. If we substitute these into $(*)$, the remaining equation is:

$c_1\vec{v}_1 = 0$, but since we know that eigenvectors are nontrivial, it must be that $c_1 = 0$. Having shown that all $c_i = 0$,

we can conclude that the k eigenvectors are linearly independent, and by induction the theorem follows. ■

Conclusion: So if we find n such distinct eigenvalues, our matrix is diagonalizable.

However, this does **NOT** mean that if you find $k < n$ distinct eigenvalues that your matrix is undiagonalizable. Rather, it means you must calculate your eigenvectors to see if your k eigenvectors *nonetheless* generate n eigenvectors.

Diagonalizing Algorithm

Upon calculating the eigenvalues and eigenvectors of **A**, if you find n linearly independent eigenvectors, then below I lay out how you can construct the equation $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where **P** is an invertible matrix, and **D** is a diagonal matrix (which is apparently similar to **A**).

- Arrange eigenvalues along the principal diagonal of an otherwise zero matrix (including non-distinct λ_ks if any).

  For example, $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

- Arrange the corresponding eigenvectors vertically as columns in a new matrix

  (in the same order as you did the λ_ks): $\mathbf{P} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}$.

- Calculate the inverse $\mathbf{P}^{-1}$. Now you have: $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, done!

Other Tidbits

Similarity is different than row equivalence.

Note from above, that the eigenvalues λ_i of **A** were necessarily the eigenvalues of the similar matrix **D**.

$$|\mathbf{D} - \lambda\mathbf{I}| = \begin{vmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0 \quad \Rightarrow \quad \lambda \in \{\lambda_1, \lambda_2\}.$$

However, consider $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Observe that you get **B** from **A** by subtracting the 1st row of **A** from the 2nd row, so they are row equivalent.

However, $|\mathbf{A} - \lambda\mathbf{I}| = \lambda^2 - 3\lambda + 1 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$, and $|\mathbf{B} - \lambda\mathbf{I}| = (1 - \lambda)^2 \Rightarrow \lambda_{1,2} = 1$. So they are not similar.

Here's a test for linear independence of vectors:

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are eigenvectors of **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of **A**.

Then **these eigenvectors are linearly independent** (see proof above).

If $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of an $n \times n$ matrix **A**, and $B_{\lambda_1} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$, $B_{\lambda_2} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s\}$, $B_{\lambda_3} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_t\}$ are the bases of the associated eigenspaces, then their union $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\} \cup \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s\} \cup \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_t\}$ is a linearly independent set of eigenvectors of **A**.

Now you are ready to predict those populations over time!

Exercises

Problem: #18 **Determine whether** $\mathbf{A} = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$ **is diagonalizable.**

If it is, find a diagonalizing matrix P and a diagonal matrix D such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Must use: $|\mathbf{A} - \lambda\mathbf{I}| = 0$,

$$\begin{vmatrix} 6-\lambda & -5 & 2 \\ 4 & -3-\lambda & 2 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1-\lambda & -5 & 2 \\ 1-\lambda & -3-\lambda & 2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -5 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda).$$

Eigenvalues: $\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

Recall: if an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, then it is diagonalizable!
(this is because $\lambda_1, \lambda_2, \lambda_3$ will necessarily produce three linearly independent eigenvectors.)

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Must use: $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = 1$: $\begin{bmatrix} 6-1 & -5 & 2 \\ 4 & -3-1 & 2 \\ 2 & -2 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow z = 0, \ y = b, \ x = b$

$$\vec{v}_1 = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ when } b = 1.$$

With $\lambda_2 = 2$: $\begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow z = c, \ y = 0, \ x = -\frac{1}{2}c$

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2}c \\ 0 \\ c \end{bmatrix} = \frac{1}{2}c\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{ when } c = 2.$$

With $\lambda_3 = 3$: $\begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 6 & -6 \\ 0 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \Rightarrow z = c, \qquad y = c, \qquad x = c.$$

$$\vec{v}_3 = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ when } c = 1.$$

So, $\mathbf{P} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$.

And, $\mathbf{A} = \mathbf{PDP^{-1}}$, where $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem: #23 Determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix P, and a diagonal matrix D such that $\mathbf{P^{-1}AP} = \mathbf{D}$.

$$\mathbf{A} = \begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Must use: $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Which is: $\begin{vmatrix} -2-\lambda & 4 & -1 \\ -3 & 5-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix}$

$$= \begin{vmatrix} 2-\lambda & 4 & -1 \\ 2-\lambda & 5-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$= (2-\lambda)(1-\lambda)^2.$

Eigenvalues: $\lambda_{1,2} = 1,\ \lambda_3 = 2.$ (diagonalizable?)

Must use: $(\mathbf{A} - \lambda_k \mathbf{I})\vec{v} = \vec{0}.$

$$\text{With } \lambda_{1,2} = 1: \quad \left.\begin{aligned} -3a + 4b - c &= 0 \\ -3a + 4b - c &= 0 \\ -a + b &= 0 \end{aligned}\right\} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{(uh oh! only one.)}$$

$$\text{With } \lambda_3 = 2: \quad \left.\begin{aligned} -4a + 4b - c &= 0 \\ -3a + 3b - c &= 0 \\ -a + b - c &= 0 \end{aligned}\right\} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(leave it to you to verify above as an exercise)

The given matrix **A** has only the two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and therefore is not diagonalizable.

Problem: #35 Let **A** be a 3×3 matrix with three distinct eigenvalues.
Tell how to construct six different invertible matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_6$
and six different diagonal matrices $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_6$ such that $\mathbf{P}_i \mathbf{D}_i (\mathbf{P}_i)^{-1} = \mathbf{A}$ for each $i = 1, 2, \ldots, 6.$

Three eigenvectors associated with three distinct eigenvalues can be arranged with six different permutations as the column vectors of the invertible matrix: $\mathbf{P} = \begin{bmatrix} \vec{v}_i & \vec{v}_j & \vec{v}_k \end{bmatrix}.$

$$\mathbf{P}_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_2 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 & \vec{v}_3 \end{bmatrix},$$

$$\mathbf{P}_4 = \begin{bmatrix} \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \end{bmatrix}, \quad \mathbf{P}_5 = \begin{bmatrix} \vec{v}_3 & \vec{v}_2 & \vec{v}_1 \end{bmatrix}, \quad \mathbf{P}_6 = \begin{bmatrix} \vec{v}_3 & \vec{v}_1 & \vec{v}_2 \end{bmatrix}.$$