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MATH 2243: **Linear Algebra** & **Differential Equations**

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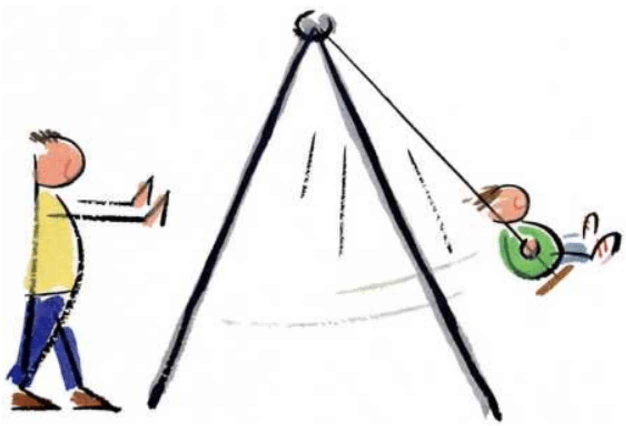
5.6: Forced Oscillations and Resonance



Forced Swing Oscillations

Forced Yo-Yo

Forced oscillation with damping:

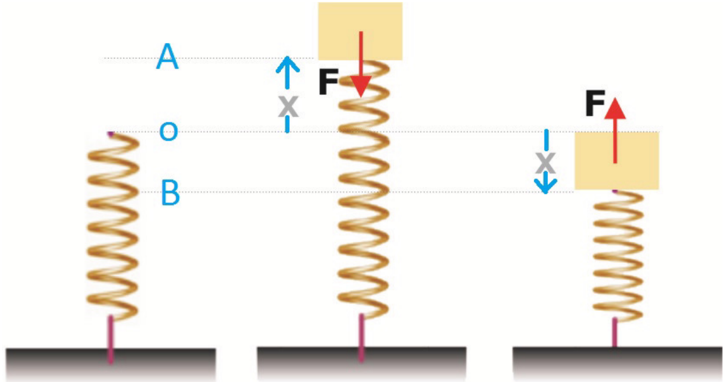
Equation: $mx'' + cx' + kx = F(t)$, where $F(t) = F_0 \cos \omega t$, or $F(t) = F_0 \sin \omega t$.

F_0 represents the amplitude of the forced oscillation, and ω is the circular frequency of the external force $F(t)$. Distinguish this from ω_0 and ω_1 which we learned about earlier being the undamped and damped circular frequencies of our non-forced system.

Forced oscillation without damping:

Equation: $mx'' + kx = F(t)$, where $F(t) = F_0 \cos \omega t$, or $F(t) = F_0 \sin \omega t$.

We now know how to calculate the complementary homogeneous solution: $x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\omega_0 = \sqrt{\frac{k}{m}}$, is the natural (undamped) frequency of the system. The resulting motion for this forced oscillation is a superposition of the oscillations ω_0 and ω.



Spring Opposing Displacement

Let's determing how the external force $F_0 \cos \omega t$ effects the Amplitude of the solution

A spring opposes static displacement, so the contribution of F_0 to the solution's amplitude is altered by the spring constant k:

Static Displacement of Spring: $x_{sd} = \frac{F_0}{k}$ results from constant force with amplitude F_0.

Also, just like when pushing a child on a swing, the more the forced oscillation ω (the push) is in sync with the system's oscillation ω_0 (the child's swinging), the more amplification (the higher the child will swing). So:

Amplification Factor: $\rho = \dfrac{1}{\left|1-\frac{\omega^2}{\omega_0^2}\right|}$.

Resonance: Is the fact that ρ can be arbitrarily large, when ω is near ω_0.

The resulting contribution to the solution's amplitude is $\rho\frac{F_0}{k}$, so the solution takes the form: $x(t) = x_c + x_p = C\cos(\omega_0 t - \alpha) \pm \rho\frac{F_0}{k}\cos\omega t$.

Observe that there is no limit on the amount of amplitude we can get for our particular solution from $\rho\frac{F_0}{k}$.

Modeling Mechanical Systems:

Kinetic Energy: $T = \frac{1}{2}m(x')^2 = \frac{1}{2}mv^2 = \int_0^v m\overline{v}\,d\overline{v}$, for movement of a mass m with velocity v. While $T = \frac{1}{2}I\omega^2$ for rotation of object w/ "momentum of inertia" I, and angular velocity ω.

Potential Energy: $V = \frac{1}{2}kx^2 = \int_0^x k\overline{x}\,d\overline{x}$, for a spring constant k displaced a distance x.

$V = mgh$, gravitational potential energy of mass m at height h above reference.

Damped Forced Oscillations: $mx''+cx'+kx = F(t)$

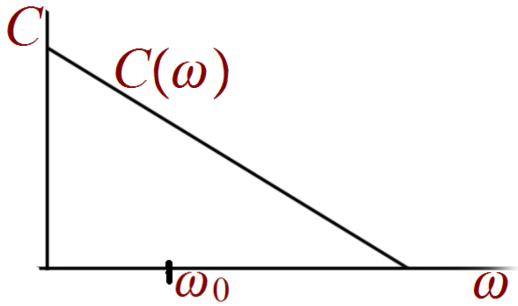
Situations: $c > c_{cr}$, $c = c_{cr}$, or $c < c_{cr}$, where $c_{cr} = \sqrt{4km}$.

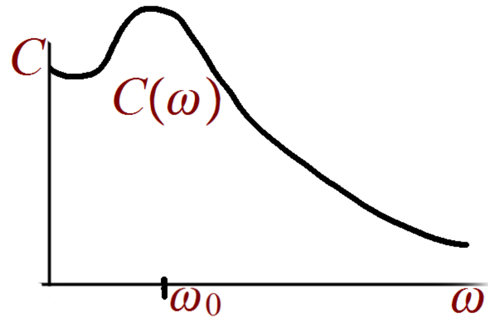
Recall, the gen. solution takes the form: $x = x_c + x_p$.

The associated complementary solution, $x_c(t)$ is a **transient solution** (notationally x_{tr}), in that it goes to zero as $t \to +\infty$. Over time, this leaves only the particular solution $x_p(t)$, which we refer to as the **steady periodic oscillation** (notationally x_{sp}).

Practical Resonance: Unlike the unbounded resonance of the undamped oscillations, the resonance of a damped oscillation (in the real world) has a finite amplitude. Similar to critical damping c_{cr} above, we have the concept of resonant damping $c_r = \sqrt{2km}$. So, If you calculate:
$x_{sp}(t) = A\cos\omega t + B\sin\omega t = C(\omega)\cos(\omega t - \alpha)$:

- ♦ For $c \geq c_r = \sqrt{2km}$, $C(\omega)$ is a steadily decreasing function of ω (not of time!!).
- ♦ For $c < c_r = \sqrt{2km}$, $C(\omega)$ has a local maximum value (*Practical Resonance*), when ω is near ω_0, then $C(\omega)$ approaches zero as $\omega \to +\infty$.





Therefore, underdamped systems with forced oscillation typically have amplitudes:

- $C(\omega)$ large if ω is close to the critical resonance frequency (ω_0).
- $C(\omega)$ close to $\frac{F_0}{k}$ (Static Displacement of Spring), if ω is small.
- $C(\omega)$ very small if ω is very large.

Videos of Disastrous Resonance: Collapsing Bridge from wind.
Shattering Glass from sound waves.

Problem: **#10** Given the equation $x'' + 3x' + 3x = 8\cos 10t + 6\sin 10t$, find the steady periodic solution: $x_{sp}(t) = C\cos(\omega t - \alpha)$.

$r^2 + 3r + 3 \Rightarrow r = \frac{-3\pm\sqrt{9-12}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$

$x_c = e^{-\frac{3}{2}t}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t\right).$

Forming Trial Solution:

$A\cos 10t + B\sin 10t$

Comparing this to the complementary solution reveals no linear dependence, so…

$x_{trial} = A\cos 10t + B\sin 10t$

$x'_{trial} = -10A\sin 10t + 10B\cos 10t, \qquad x''_{trial} = -100A\cos 10t - 100\sin B10t$

$$\begin{aligned}
&x''_{trial} + 3x'_{trial} + 3x_{trial} \\
&\quad= (-100A\cos 10t - 100B\sin 10t) + 3(-10A\sin 10t + 10B\cos 10t) + 3(A\cos 10t + B\sin 10t) \\
&\quad= (-100A\cos 10t - 100B\sin 10t) + (-30A\sin 10t + 30B\cos 10t) + (3A\cos 10t + 3B\sin 10t) \\
&\quad= (30B - 100A + 3A)\cos 10t + (3B - 100B - 30A)\sin 10t \\
&\quad= (30B - 97A)\cos 10t + (-97B - 30A)\sin 10t \\
&\quad\stackrel{?}{=} 8\cos 10t + 6\sin 10t.
\end{aligned}$$

Comparing sides of the equation:

$30B - 97A = 8, \text{ and } -97B - 30A = 6$

$B = \frac{8}{30} + \frac{97}{30}A, \qquad -97(\frac{4}{15} + \frac{97}{30}A) - 30A = -\frac{10309}{30}A - \frac{388}{15} = 6,$

$\qquad A = -\frac{30\left(6+\frac{388}{15}\right)}{10309} = -\frac{956}{10309}.$

$B = \frac{8}{30} + \frac{97}{30}(-\frac{956}{10309}) = -\frac{342}{10309}.$

$x_{sp}(t) = -\frac{956}{10309}\cos 10t - \frac{342}{10309}\sin 10t$ $\qquad$ Are we done?

"Find the steady periodic solution: $x_{sp}(t) = C\cos(\omega t - \alpha)$."

$$C = \sqrt{(-\frac{956}{10309})^2 + (-\frac{342}{10309})^2} = \sqrt{(\frac{913936}{106275481}) + (\frac{116964}{106275481})} = \sqrt{\frac{100}{10309}} = \frac{10\sqrt{61}}{793}.$$

$x_{sp}(t) = \frac{10\sqrt{61}}{793}\cos(10t - \alpha).$

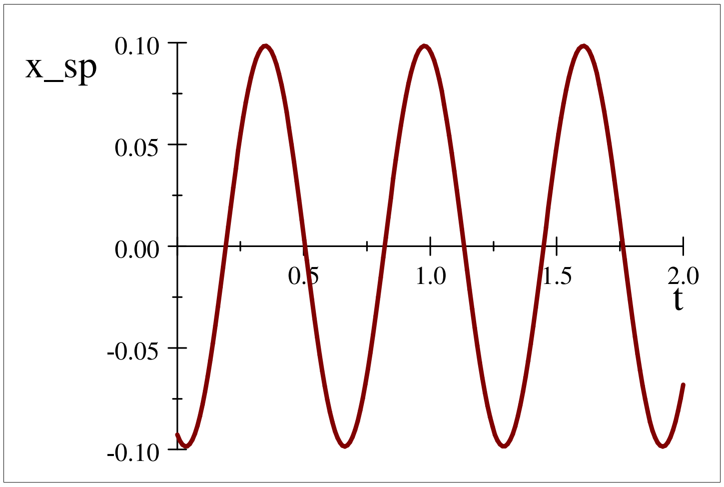
where $\alpha = (???) + \tan^{-1}\frac{B}{A}$

$\frac{B}{A} = \frac{-\frac{342}{10309}}{-\frac{956}{10309}} = \frac{171}{478}$

Recall: $\alpha = \begin{cases} \tan^{-1}\frac{B}{A} & \text{if } A, B > 0 \text{ (1st quadrant)}, \\ \pi + \tan^{-1}\frac{B}{A} & \text{if } A < 0 \text{ (2nd/3rd quadrant)}, \\ 2\pi + \tan^{-1}\frac{B}{A} & \text{if } A > 0, B < 0 \text{ (4th quadrant)}. \end{cases}$

$\alpha = \pi + \tan^{-1}\frac{171}{478} \approx 3.4851.$ $\qquad$ (third quadrant angle)

$x_{sp}(t) \approx \frac{10\sqrt{61}}{793}\cos(10t - 3.485).$



$$\frac{10\sqrt{61}}{793}\cos(10t - 3.485)$$

Problem: **#13** Find and plot the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ of the differential equation $x'' + 2x' + 26x = 600\cos 10t$, with initial conditions $x(0) = 10$, and $x'(0) = 0$. Also, find and plot the actual solution: $x(t) = x_{tr}(t) + x_{sp}(t) = x_c(t) + x_p(t)$.

$$r^2 + 2r + 26 = 0, \qquad r = \frac{-2 \pm \sqrt{4-104}}{2} = -1 \pm 5i.$$

$$x_c(t) = x_{tr}(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$$

$$x_0 = x_{trial} = A\cos 10t + B\sin 10t$$
$$x'_{trial} = -10A\sin 10t + 10B\cos 10t$$
$$x''_{trial} = -100A\cos 10t - 100B\sin 10t$$

$$x''_{trial} + 2x'_{trial} + 26x_{trial} =$$
$$(-100A\cos 10t - 100B\sin 10t) + 2(-10A\sin 10t + 10B\cos 10t) + 26(A\cos 10t + B\sin 10t)$$
$$= (-100A\cos 10t - 100B\sin 10t) + (-20A\sin 10t + 20B\cos 10t) + (26A\cos 10t + 26B\sin 10t)$$
$$= (20B - 100A + 26A)\cos 10t + (26B - 100B - 20A)\sin 10t$$
$$= (20B - 74A)\cos 10t + (-74B - 20A)\sin 10t$$
$$\stackrel{?}{=} 600\cos 10t.$$

Comparing sides of the equation:

$$20B - 74A = 600, \qquad -74B - 20A = 0.$$
$$B = -\frac{20}{74}A, \qquad 20\left(-\frac{20}{74}A\right) - 74A = -\frac{2938}{37}A = 600, \qquad A = -\frac{600 \cdot 37}{2938} = -\frac{11\,100}{1469}$$

$$B = -\frac{20}{74}\left(-\frac{11\,100}{1469}\right) = \frac{3000}{1469}.$$

So, $x_{sp}(t) = -\frac{11100}{1469}\cos 10t + \frac{3000}{1469}\sin 10t.$ Now what?

Putting it in the right form:

$$C = \sqrt{\left(-\frac{11100}{1469}\right)^2 + \left(\frac{3000}{1469}\right)^2} = \frac{300}{\sqrt{1469}}.$$

$$x_{sp}(t) = \frac{300}{\sqrt{1469}}\cos(10t - \alpha).$$

Determining α :

$$\frac{B}{A} = \frac{\frac{3000}{1469}}{-\frac{11100}{1469}} = -\frac{10}{37}$$

$$\alpha = \pi + \tan^{-1}\frac{10}{-37} \approx 2.88 \qquad \text{(second quadrant angle)}$$

So, $x_{sp}(t) \approx \frac{300}{\sqrt{1469}}\cos(10t - 2.88).$

Gen. solution: $x(t) = x_{tr}(t) + x_{sp}(t) = e^{-t}(c_1\cos 5t + c_2\sin 5t) + \frac{300}{\sqrt{1469}}\cos(10t - 2.8776).$

Are we done?

"Find the actual solutions $x(t) = x_{sp}(t) + x_{tr}(t)$ that satisfies the given initial conditions.

$x(0) = 10, \qquad x'(0) = 0.$"

To keep the accuracy, and not deal with decimals, I revert back to the previous version for this…

$$x(t) = x_{tr} + x_{sp}(t) = e^{-t}(c_1\cos 5t + c_2\sin 5t) - \frac{11100}{1469}\cos 10t + \frac{3000}{1469}\sin 10t$$

$$x'(t) = [e^{-t}(5c_2\cos 5t - 5c_1\sin 5t) - e^{-t}(c_1\cos 5t + c_2\sin 5t)] + \frac{111000}{1469}\sin 10t + \frac{30000}{1469}\cos 10t$$

$$x'(0) = 0 = [e^0(5c_2\cos 0 - 5c_1\sin 0) - e^0(c_1\cos 0 + c_2\sin 0)] + \frac{111000}{1469}\sin 0 + \frac{30000}{1469}\cos 0$$

$$= [(5c_2) - (c_1)] + \frac{30000}{1469}$$

$$x(0) = 10 = e^0(c_1\cos 0 + c_2\sin 0) - \frac{11100}{1469}\cos 0 + \frac{3000}{1469}\sin 0 = c_1 - \frac{11100}{1469}$$

$$c_1 = \frac{11100}{1469} + 10 = \frac{25\,790}{1469}, \qquad c_2 = \frac{1}{5}c_1 - \frac{1}{5}\frac{30000}{1469} = \frac{1}{5}\frac{25\,790}{1469} - \frac{1}{5}\frac{30000}{1469} = -\frac{842}{1469}.$$

Therefore, $x(t) = e^{-t}\left(\frac{25\,790}{1469}\cos 5t - \frac{842}{1469}\sin 5t\right) + \frac{300}{\sqrt{1469}}\cos(10t - 2.88).$

We done?

We could convert the complementary solution into the form: $x_{tr}(t) = C\cos(\omega t - \alpha)$.

$$x_{tr}(t) = \frac{1}{1469}e^{-t}(25790\cos 5t - 842\sin 5t)$$

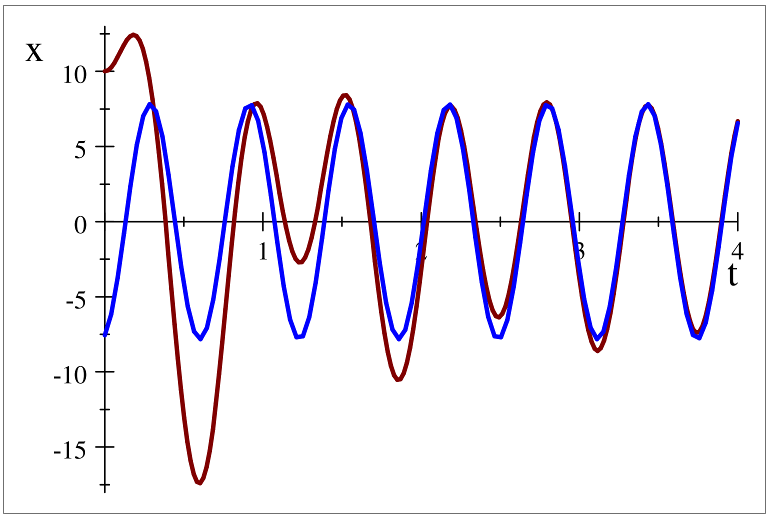
$$C = \sqrt{25790^2 + (-842)^2} = 2\sqrt{166458266}$$

$$x_{tr}(t) = \frac{2\sqrt{166458266}}{1469}e^{-t}\cos(5t - \beta)$$

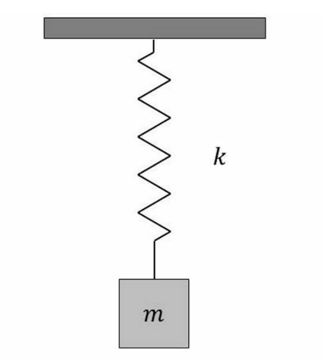
$$\frac{B}{A} = \frac{-842}{25790} = -\frac{421}{12895}$$

$$\beta = 2\pi + \tan^{-1}\frac{-421}{12895} \approx 6.2505. \qquad \text{(fourth quadrant angle)}$$

$$x(t) \approx \frac{2\sqrt{166458266}}{1469}e^{-t}\cos(5t - 6.2505) + \frac{300}{\sqrt{1469}}\cos(10t - 2.8776)$$



x (red), and x_{sp} (blue)



Problem 19: A mass weighing 100 pounds (mass $m = 3.125$ slugs in *fps* units) is attached to the end of a spring that is stretched 1 inch, by a force of 100 pounds (duh!). Another force ($F_0\cos\omega t$) acts on the mass. At what frequency in hertz (cycles/sec) will resonant oscillations occur? Neglect damping.

Resonant frequency is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{3.125}}$.

$m = 3.125$ slugs (*fps*), and $k = \frac{Force}{Distance} = \frac{100\,lbs}{\frac{1}{12}\,ft} = 1200\ lb/ft$.

So the resonant frequency is $\omega_0 = \sqrt{\frac{1200}{3.125}}$ (units?)

$$= \sqrt{384}\,\frac{rad}{\sec} \approx 19.6\,\frac{rad}{\sec}.$$

\# of $Hz = \frac{\omega_0}{2\pi} = \frac{\sqrt{384}}{2\pi} \approx 3.12\ Hz$.

So, resonance will occur when $\frac{\omega}{2\pi} \approx \frac{\omega_0}{2\pi} = 3.12\ Hz$.

If $c = 12$, and $\omega = 90$, what behavior should we expect from the amplitude?

Resonant damping: $c_r = \sqrt{2km} = \sqrt{2 \cdot 1200 \cdot 3.125} \approx 86.6$.

Recall $\omega_0 \approx 19.6$, and "for $c < \sqrt{2km}$, $C(\omega)$ reaches a maximum value (*Practical Resonance*), at some value of $\omega < \omega_0$, then it approaches zero as $\omega \to +\infty$."

We should expect the amplitude to be less than the practical resonance.

If $c = 200$ and ω is very large, what behavior should we expect from the amplitude?

Recall: ♦ For $c \geq \sqrt{2km}$, $C(\omega)$ is a steadily decreasing function of ω.

So the amplitude should be small.

If $c = 12$, and $\omega < 20$, what behavior should we expect from the amplitude?

The amplitude may be at its peak (practical resonance), or at least not very small.

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