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5.1: Second-Order Linear Equations:

Modeling the world with first-order DEQs $y' + q(x)y = f(x)$ assumes a simple situation in which the coefficient in front of y'' is zero (and similarly with $y''', y^{(4)}$, etc.).

We have seen that solving a first order DEQ gives us a 1-dimensional family of solutions (e.g., $y = Ce^x$). But if we assume a more complicated scenario where the coefficient in front of y'' is nonzero, we generate more solutions. Solving this 2nd order DEQ (e.g., $y'' + p(x)y' + q(x)y = f(x)$) gives us a 2-dimensional family of solutions (e.g., $y = Ae^x + Be^{-x}$).

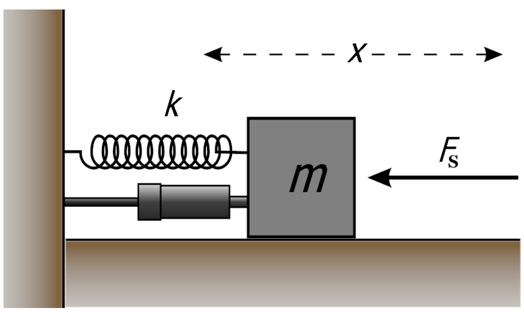
Linear DEQ: $e^x y'' + \cos(x)y' + (1 + \sqrt{x})y = \tan^{-1}(x)$

Non-linear DEQ: $y'' + 3(y')^2 + 4y^3 = 0$

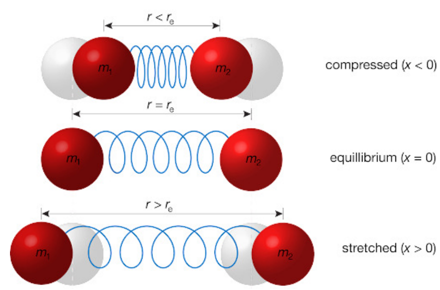
Non-homogenous: $x^2y'' + 2xy' + 3y = \cos x$,

which is **associated** with **homogenous** DEQ: $x^2y'' + 2xy' + 3y = 0$.

Mechanical Systems



Mass, Spring, Damper (see animation in class)

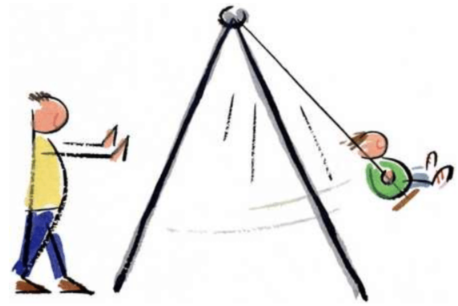


Hooke's Law: $F_S = -kx$, where $k > 0$ (Spring Force)

$F_R = -cv = -c\frac{dx}{dt}$, where $c > 0$ (Resistance/Damping Force)

Newton: $F = ma = m\frac{d^2x}{dt^2}$.

Therefore $m\frac{d^2x}{dt^2} = F_S + F_R$, or $mx'' + cx' + kx = 0$. (homogeneous model with damping)



External Periodic Force

$mx'' + cx' + kx = F(t)$ (model which includes damping and nonhomogeneous external force)

More General Mathematical Treatment

$A(t)x'' + B(t)x' + C(t)x = F(t)$ or $A(x)y'' + B(x)y' + C(x)y = F(x)$.

Normal Form: $y'' + p(x)y' + q(x)y = f(x)$, obtained if $A(x) \neq 0$ on the interval of interest.

Superposition of Homogeneous DEQ Solutions Theorem: If y_1, y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$, and c_1, c_2 are constants, then $y = c_1 y_1 + c_2 y_2$ is also a solution.

This generalizes to nth order DEQs.

Proof: We are given that $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$.

We must show that $y'' + p(x)y' + q(x)y = 0$ when substituting in $y = c_1 y_1 + c_2 y_2$.

Observe $y' = c_1 y_1' + c_2 y_2'$ and $y'' = c_1 y_1'' + c_2 y_2''$.

Substituting in: $y'' + p(x)y' + q(x)y = (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2)$

$= c_1 y_1'' + c_1 p(x) y_1' + c_1 q(x) y_1 + c_2 y_2'' + c_2 p(x) y_2' + c_2 q(x) y_2$

$= c_1 (y_1'' + p(x)y_1' + q(x)y_1) + c_2 (y_2'' + p(x)y_2' + q(x)y_2)$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$ ■

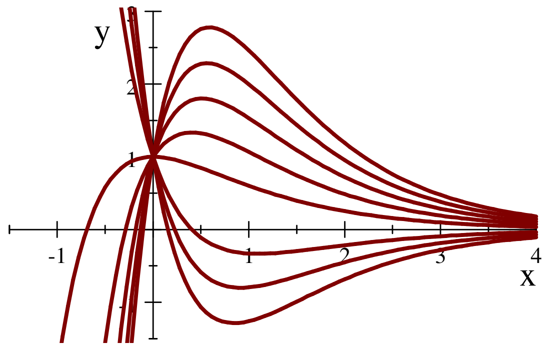
So solutions to homogeneous DEQs form a vector space.

Existence and Uniqueness for Linear DEQs Theorem: Given: $y'' + p_1(x)y' + p_2(x)y = f(x)$,
if at a point $x = a$ the expressions $p_1(x)$, $p_2(x)$, and $f(x)$ are continuous on some interval,
then there's a **unique** solution to $y'' + p_1(x)y' + p_2(x)y = f(x)$ on that interval satisfying **initial conditions**:
$y(a) = b_0$, $y'(a) = b_1$, for any $b_0, b_1 \in \mathbb{R}$.

This generalizes to nth order DEQs.

Recall that each first order DEQ gave a unique solution for each point (a, b).

For the 2nd order DEQ above, note that if we choose initial condition $y(a) = b_0$, there is still an infinite number of solutions based upon our choice of initial condition $y'(a) = b_1$. Geometrically, this means that for every point in the plane, we can choose any (finite) slope we want, and there will be a solution going through that point with that slope.



$y''+3y'+2y = 0$ with $y(0) = 1$, but different slopes

Recall that in $\mathbb{R}^2$, we needed two linearly independent vectors to span the vector space. Similarly, to span the solution set $\mathbb{S}$ of a homogeneous second order DEQ, you need two linearly independent vectors, which in our case are functions y_1, y_2.

And as with the vectors in $\mathbb{R}^2$, y_1, y_2 are **linearly independent** if they are not constant multiples of each other.

$y_1 \neq k y_2$ where $k \in \mathbb{R}$.

However, it's not always clear cut whether two functions are constant multiples of each other.

Independence of Functions

Let's develop the condition under which, given a DEQ $y'' + p_1(x)y' + p_2(x)y = f(x)$, and any two solutions y_1, y_2 we can say that the solution $y = c_1 y_1 + c_2 y_2$ represents the general solution (spans the solutions space).

Well we know y_1, y_2 must be linearly independent, but how can we check this?

Recall that our existence theorem above for DEQ solutions suggests that if y IS our general solution, we should be able to uniquely find any particular solution using any initial conditions $y(a) = b_0, y'(a) = b_1$. In other words, solve the system:

$$\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) &= b_1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Observe that we can solve for c_1, c_2 uniquely if the determinant of the matrix is nonzero.

Further observe that for y to be the general solution, the determinant must be nonzero for every choice of a.

This suggests a method for identifying functions which are linearly independent.

Wronskian (denoted by: W):

Given $y_1(x),\ y_2(x)$ we denote $W(x) = W(y_1, y_2) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$

This generalizes to n EQs with an $n \times n$ determinant with $n-1$ derivatives.

Wronskian of Solutions Theorem: If y_1, y_2 are solutions to $y'' + p_1(x)y' + p_2(x)y = 0$ on an interval I where p_1, p_2 are continuous, then:

- ♦ y_1, y_2 are linearly dependent if and only if $W(y_1, y_2) = 0,$ at each point x in I.
- ♦ y_1, y_2 are linearly independent if and only if $W(y_1, y_2) \neq 0$ at each point x in I.

For all other solutions $y(x)$ to the homogeneous DEQ,

there exists $c_1, c_2 \in \mathbb{R}$ such that: $y(x) = c_1 y_1(x) + c_2 y_2(x)$ (**general solution**).

This generalizes to nth order DEQs with $p_1, \ldots, p_n,$ with n solutions $y_1, \ldots, y_n,$ and $c_1, \ldots, c_n \in \mathbb{R}$.

General Solutions of Homogeneous DEQs Theorem: Let y_1, y_2 be linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$, with p, q continuous on some interval I. If Y is any solution whatsoever on I, then there exists $c_1, c_2 \in \mathbb{R}$ such that $Y(x) = c_1 y_1 + c_2 y_2$, for all x on I.

Proof: Choose $a \in I$. Consider:

$$\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= Y(a), \\ c_1 y_1'(a) + c_2 y_2'(a) &= Y'(a). \end{aligned}$$

$$\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y(a) \\ Y'(a) \end{bmatrix}$$

Observe: $W(y_1, y_2) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$ (for all x, including $x = a$), since we have independence.

So we can reduce the system to solve for c_1, c_2. But does $Y = c_1 y_1 + c_2 y_2$ for the rest of $x \in I$?

Define $G(x) := c_1 y_1(x) + c_2 y_2(x)$.
Observe that this solves the DEQ since it is a linear combination of solutions.

Recall that the uniqueness theorem tells us that solutions which satisfy

initial conditions $y(a) = b_1$ and $y'(a) = b_2$ are unique.

Note that $G(a) = c_1 y_1(a) + c_2 y_2(a) = Y(a)$.

and $G'(a) = c_1 y_1'(a) + c_2 y_2'(a) = Y'(a)$.

So, since both G and Y satisfy the same initial conditions, and are both solutions to the DEQ, $Y(x) = G(x)$, on I.

And we have $Y(x) = c_1 y_1 + c_2 y_2$, for all x on I. ■

2nd Order Homogeneous DEQs w/ Constant Coefficients

In general, it is difficult/impossible to solve 2nd order DEQs.

So let us simplify to *linear* 2nd order homogeneous DEQs with constant coefficients.

$$ay'' + by' + cy = 0$$

To solve this, we need a y such that a linear combination of its derivatives is equal to y multiplied by a constant (i.e., $ay'' + by' = -cy$).

Note if $y := e^{rx}$, then: $y' = (e^{rx})' = re^{rx} = ry$. And $y'' = r^2 y$.

This implies we might be able to make this type of substitution to find a solution, by solving for r.

$$ar^2 y + bry + cy = 0$$
$$ar^2 + br + c = 0, \text{ for } y \neq 0.$$

Characteristic Equation Algorithm: To solve $ay'' + by' + cy = 0$, replace y'', y', y with $r^2, r, 1$.

Then, algebraically solve the characteristic equation ($ar^2 + br + c = 0$) for r.

- ♦ If solutions r_1, r_2 are real & distinct, $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the **general solution** to our DEQ, and the solution space has basis $\left\{e^{r_1 x},\ e^{r_2 x}\right\}$.

- ♦ If $r_1 = r_2$, then $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ is the general solution, and the solution space has basis $\left\{e^{r_1 x},\ x e^{r_1 x}\right\}$.

This generalizes to nth order DEQs with $y^{(n)}, \ldots y', y$ and $r^n, \ldots r^2, r, 1$.

Exercises

Problem: **#36** Find the general solution: $2y'' + 3y' = 0$.

$2r^2 + 3r = 0$.

$r(2r + 3) = 0$.

$r = 0,\ -\frac{3}{2}$;

$y(x) = c_1 + c_2 e^{-\frac{3}{2}x}.$

Problem: **#44** Given the general solution $y(x) = c_1 e^{10x} + c_2 e^{-10x}$ of a homogeneous second order DEQ, find the DEQ in the form $ay'' + by' + cy = 0$ with constant coefficients.

$(r-10)(r+10) = 0$

$r^2 - 100 = 0.$

$y'' - 100y = 0.$

Problem: **#31** $y_1 = \sin x^2$ and $y_2 = \cos x^2$ are linearly independent functions, but show that their Wronskian vanishes (is equal to zero) at $x = 0$. Why does this imply that there is no differential equation having both y_1 and y_2 as (global) solutions, of the form $y'' + p_1(x)y' + p_2(x)y = 0$, with both p_1 and p_2 continuous everywhere?

$$W(y_1, y_2) = \begin{vmatrix} \sin x^2 & \cos x^2 \\ 2x\cos x^2 & -2x\sin x^2 \end{vmatrix}$$

$$= -2x\sin^2 x^2 - 2x\cos^2 x^2$$

$$= -2x(\sin^2 x^2 + \cos^2 x^2) = -2x.$$

$-2x$ vanishes at $x = 0$.

"**Why does this imply that there is no differential equation of the form** $y'' + p_1(x)y' + p_2(x)y = 0$, **with both** p_1 **and** p_2 **continuous everywhere**, **having both** y_1 **and** y_2 **as global solutions**?"

In order for y_1 and y_2 to be linearly **independent** solutions of the equation $y'' + p_1 y' + p_2 y = 0$

(with p_1 and p_2 both continuous) on an open interval I containing $x = 0$,

the **Wronskian of Solutions Theorem** requires $W \neq 0$ on I.