MATH 2243: **Linear Algebra** & **Differential Equations**

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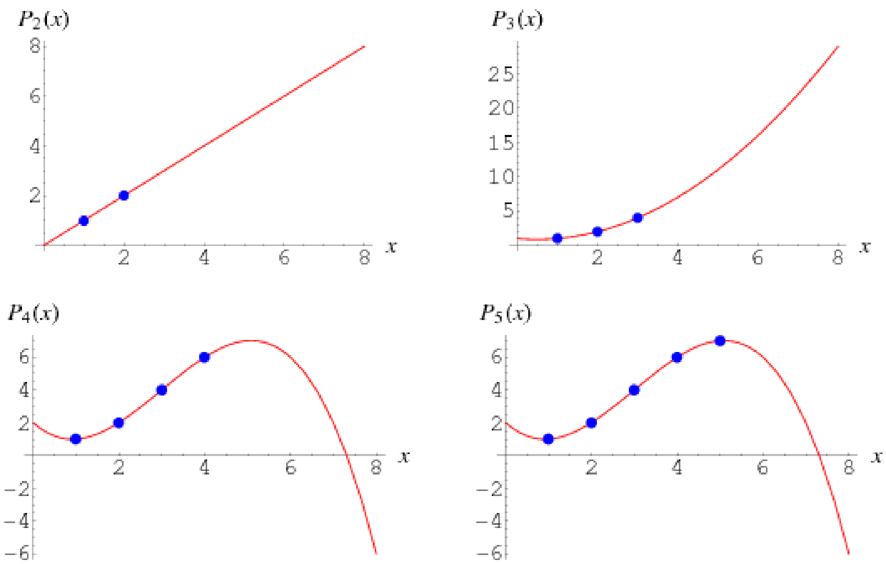
3.7: Linear Equations and Curve Fitting



Given a finite number of data points, how do we come up with a curve which best represents the data (illustrated above)? One method is to form a polynomial of degree n :
$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where the a_i are constants (for example $f(x) = 7 + 2x$).
Interpolating Polynomial: The **unique** nth degree polynomial that fits the $n+1$ given data points.

How do discover such a polynomial? Well, for each data point (x_i, y_i), we require that the polynomial pass through it, so we require that $f(x_i) = y_i$. And if we do this for $n+1$ data points, we end up with a system:

$$\begin{aligned} a_0 + a_1x_0 + a_2(x_0)^2 + \ldots + a_n(x_0)^n &= y_0, \\ a_0 + a_1x_1 + a_2(x_1)^2 + \ldots + a_n(x_1)^n &= y_1, \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_0 + a_1x_n + a_2(x_n)^2 + \ldots + a_n(x_n)^n &= y_n. \end{aligned}$$

Recall that the (x_i, y_i) have been given to us (these are just numbers), therefore the above system is just a linear system of $n+1$ equations in $n+1$ unknowns (the a_i are our unknowns). So we can solve this using our matrix method. Putting this in a matrix equation, we have:

$$A \cdot \vec{a} = \vec{y}, \text{ where } a = [a_0, a_1, \ldots, a_n]^T,\ y = [y_0, y_1, \ldots, y_n]^T, \text{ and } A = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}.$$

You may recall that this matrix is called the **Vandermonde matrix**, which has the special property that if all of the $x_0, \ldots, x_n$ are unique (i.e., the graph passes the vertical line test), then the matrix A is nonsingular (i.e., the determinant is not zero). But why do we care? Because by Theorem 7 in section 3.5, we can now say that the system has a **unique** solution when solving for the coefficients a_i. In other words, there is a unique nth degree polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ that fits the $n+1$ data points we were given. This **unique** polynomial is called the **interpolating polynomial**.

Problem: #1 Find the 1st degree polynomial $y = f(x)$ that fits the points: $(1,1)$ and $(3,7)$.

1st degree polynomial ansatz: $y(x) = a + bx$.

So we need to form the system of equations $A \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Or in our case:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Reducing $[A|\vec{y}]$: $$\begin{bmatrix} 1 & 1 & | & 1 \\ 0 & 2 & | & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & 3 \end{bmatrix}$$

$\Rightarrow \quad a = -2, \ b = 3.$

So, $y(x) = -2 + 3x$.

Problem: #4 Find the 2nd degree polynomial $y = f(x)$ that fits the points: $(-1,1)$, $(1,5)$, and $(2,16)$.

2nd degree polynomial ansatz: $y(x) = a + bx + cx^2$.

So we need to form the system of equations $A \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}.$$

Or in our case:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 16 \end{bmatrix}$$

Reducing $[A|\vec{y}]$: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 3 & 3 & 15 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 3 & 15 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

$\Rightarrow \quad a = 0, \ b = 2, \ c = 3.$

So, $y(x) = 2x + 3x^2$.

Problem: #7 Find the 3rd degree polynomial $y = f(x)$ that fits the points: $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, -4)$.

3rd degree polynomial ansatz: $y(x) = a + bx + cx^2 + dx^3$.

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

$\Rightarrow \quad a = 0, \ b = \frac{4}{3}, \ c = 1, \ d = -\frac{4}{3}.$

So, $y(x) = \frac{1}{3}(4x + 3x^2 - 4x^3)$.