MATH 2243: Linear Algebra & Differential Equations

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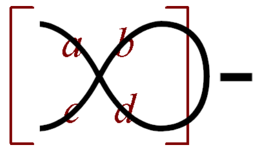
3.6: Determinants

$$\det \mathbf{A} = |\mathbf{A}| = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



Determiniants of $\mathbf{n} \times \mathbf{n} = \mathbf{3} \times \mathbf{3}$ Matrices:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$, and similarly for $n \geq 2$.

Below, we will make use of a "checkerboard" of signs: $\begin{vmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \vdots \\ \vdots & \vdots & \dots & \ddots \end{vmatrix}$.

a_{ij}th **Minor** M_{ij}: found by ignoring the ith row and the jth column,

and taking the determinant of the remaining elements. For example:

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, and **Cofactor**: $A_{ij} := \big(\text{Checkerboard Sign}\big)M_{ij}$.

So, $A_{12} = (-1)M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Therefore, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

Notice how easy these calculations would be if some of the $a_{ij} = 0$.

Later, we'll find out how to arrange for this to happen.

Definition: Most generally, the determinant ($\det \mathbf{A} = |a_{ij}|$) of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined as:

$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$, where A_{ij} is the ijth cofactor.

Above, we expanded our determinant along the top row of the original matrix. However, you can calculate the determinant using any row or column. And indeed, if there is a row or column with entries that are zero (or that you can manipulate to be zero), it is strategic to use that row or column to simplify your determinant calculation.

Corollary: If $\mathbf{A}$ contains either a zero column or row vector, then $|\mathbf{A}| = 0$.

This follows since we can choose the zero row or column to do our expansion, and the resulting expression will have cofactor coefficients of zero.

Important Property: $\mathbf{A}$ is invertible if $|\mathbf{A}| \neq 0$.

Transpose:

If $\mathbf{A} = \begin{bmatrix} \mathbf{c} & \mathbf{a} & \mathbf{t} \\ p & e & n \\ \mathbf{m} & \mathbf{o} & \mathbf{m} \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} \mathbf{c} & p & \mathbf{m} \\ \mathbf{a} & e & \mathbf{o} \\ \mathbf{t} & n & \mathbf{m} \end{bmatrix}$.

Transpose Properties:

- ♦ $(\mathbf{A}^T)^T = \mathbf{A}$
- ♦ $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
- ♦ $(c\mathbf{A})^T = c\mathbf{A}^T$
- ♦ $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ (recall: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$)
- ♦ $\det(\mathbf{A}^T) = \det \mathbf{A}$

Determinant Properties:

♦ $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$, or equivalently, $\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$.

As a direct result of the above property, we prove $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. (recall $\mathbf{A}$ invertible implies $|\mathbf{A}| \neq 0$)

Proof: Observe that the result assumes $\mathbf{A}^{-1}$ and $\mathbf{A}$ exist, and therefore we can say:

$\mathbf{AA}^{-1} = \mathbf{I}$. Using the determinant property on this equation:

$|\mathbf{AA}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}| = |\mathbf{I}| = 1$. And dividing by $|\mathbf{A}|$ gives us:

$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. ■

Calculating the Determinant More Easily:

For the following column and row manipulations, assume k is some constant.

Pull constants from individual rows or columns: (Bolded text is to indicate changes.)

$$\begin{vmatrix} \mathbf{ka}_{11} & \mathbf{ka}_{12} & \mathbf{ka}_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathbf{k}\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} a_{11} & \mathbf{ka}_{12} & a_{13} \\ a_{21} & \mathbf{ka}_{22} & a_{23} \\ a_{31} & \mathbf{ka}_{32} & a_{33} \end{vmatrix} = \mathbf{k}\begin{vmatrix} a_{11} & \mathbf{a}_{12} & a_{13} \\ a_{21} & \mathbf{a}_{22} & a_{23} \\ a_{31} & \mathbf{a}_{32} & a_{33} \end{vmatrix}.$$

Proof for $n = 3$:

Switch columns or rows by changing sign.

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & a_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & a_{23} \\ \mathbf{a}_{31} & \mathbf{a}_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} \mathbf{a}_{12} & \mathbf{a}_{11} & a_{13} \\ \mathbf{a}_{22} & \mathbf{a}_{21} & a_{23} \\ \mathbf{a}_{32} & \mathbf{a}_{31} & a_{33} \end{vmatrix}.$$

Proof for $n = 3$:

Identical rows or columns means determinant is zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b & c & d \\ b & c & d \end{vmatrix} = 0, \text{ and } \begin{vmatrix} a_{11} & a & a \\ a_{21} & b & b \\ a_{31} & c & c \end{vmatrix} = 0.$$

Proof for $n = 3$:

Can add a multiple of a row to another row, and can do the same with columns.

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ b & c & d \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{11}+\mathbf{kb} & \mathbf{a}_{12}+\mathbf{kc} & \mathbf{a}_{13}+\mathbf{kd} \\ b & c & d \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} \mathbf{a}_{11} & a_{12} & b \\ \mathbf{a}_{21} & a_{22} & c \\ \mathbf{a}_{31} & a_{32} & d \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{11}+\mathbf{kb} & a_{12} & b \\ \mathbf{a}_{21}+\mathbf{kc} & a_{22} & c \\ \mathbf{a}_{31}+\mathbf{kd} & a_{32} & d \end{vmatrix}.$$

Proof for $n = 3$:

Determinant of (**upper or lower**) **triangular matrices is the product of the main diagonal**.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33}. \text{ Therefore } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = 0.$$

Proof for $n = 3$:

Cramer's Rule:

Solution to $\mathbf{A}\vec{x} = \vec{b}$.

Given: $\mathbf{A}^{n \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with $|\mathbf{A}| \neq 0$, and given $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, Define: $|\mathbf{B}_1| := \begin{vmatrix} \mathbf{b}_1 & a_{12} & a_{13} \\ \mathbf{b}_2 & a_{22} & a_{23} \\ \mathbf{b}_3 & a_{32} & a_{33} \end{vmatrix}$, $|\mathbf{B}_2| := \begin{vmatrix} a_{11} & \mathbf{b}_1 & a_{13} \\ a_{21} & \mathbf{b}_2 & a_{23} \\ a_{31} & \mathbf{b}_3 & a_{33} \end{vmatrix}$, $|\mathbf{B}_3| := \begin{vmatrix} a_{11} & a_{12} & \mathbf{b}_1 \\ a_{21} & a_{22} & \mathbf{b}_2 \\ a_{31} & a_{32} & \mathbf{b}_3 \end{vmatrix}$.

The solution to $\mathbf{A}\vec{x} = \vec{b}$ is $\vec{x} = \langle x_1, x_2, x_3 \rangle$, where:

$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|}$, $x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|}$, $x_3 = \frac{|\mathbf{B}_3|}{|\mathbf{A}|}$, and similarly with matrices sized $n > 3$.

Matrix Inverse $\mathbf{A}^{-1}$ when $n > 3$:

In addition to the $\left[\mathbf{A} \mid I\right] \to \left[I \mid \mathbf{A}^{-1}\right]$ method, we have:

If $\mathbf{A}$ is invertible (so first check the determinant), then we can first construct a **cofactor** matrix $[A_{mn}]$, which consists of the **cofactors** (NOT the components) of $\mathbf{A}$. And then: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$. (notice how we took the transpose)

Proof: We need to solve $\mathbf{AX} = \mathbf{I}$ for $\mathbf{X}$. So, for each column vector $\vec{x}_i$ is matrix $\mathbf{X}$, use cramer's rule to solve $\mathbf{A}\vec{x}_i = \mathbf{e}_i$.
■

For inverse of a 3×3 matrix, if $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[A_{mn}]^T = \frac{1}{\det \mathbf{A}}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{\mathbf{T}} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

... where A_{mn} are the cofactors (NOT the components of $\mathbf{A}$!).

Video Tutorials (visually rich and intuitive): https://youtu.be/uQhTuRlWMxw
https://youtu.be/Ip3X9LOh2dk

Exercises

Problem: **#15** Use the method of elimination (*determinant* row/column operations) to evaluate the determinant.

$$\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$\Rightarrow \quad c_2 + (-4c_1) \quad \Rightarrow \quad \begin{vmatrix} -2 & 13 & 4 \\ 5 & -17 & 1 \\ 1 & 0 & 5 \end{vmatrix} \quad \Rightarrow \quad c_3 + (-5c_1) \quad \Rightarrow \quad \begin{vmatrix} -2 & 13 & 14 \\ 5 & -17 & -24 \\ 1 & 0 & 0 \end{vmatrix}$$

(note that the above operations can only be performed on determinants, not matrices!)

$$\Rightarrow \quad -\begin{vmatrix} -2 & 13 & 14 \\ 1 & 0 & 0 \\ 5 & -17 & -24 \end{vmatrix} \quad \Rightarrow \quad +\begin{vmatrix} 1 & 0 & 0 \\ -2 & 13 & 14 \\ 5 & -17 & -24 \end{vmatrix}$$

(don't need these steps here, but thought I'd show row-swapping/sign-changes)

$$= +1\begin{vmatrix} 13 & 14 \\ -17 & -24 \end{vmatrix} = -74. \qquad \text{(Checkerboard and Fish)}$$

Problem: **#31** Use Cramer's rule to solve the system:

$$\begin{aligned} 2x_1 - 5x_3 &= -3 \\ 4x_1 + 3x_3 - 5x_2 &= 3 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$\mathbf{A}\vec{x} = \vec{b}$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} \qquad \text{(let's further simplify the column with the zero in it)}$$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ -6 & 0 & 8 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= 1(16 - 30) = 14 \neq 0. \qquad \checkmark$$

$$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} \mathbf{-3} & 0 & -5 \\ \mathbf{3} & -5 & 3 \\ \mathbf{1} & 1 & 1 \end{vmatrix} = -\frac{8}{7}, \qquad x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} 2 & \mathbf{-3} & -5 \\ 4 & \mathbf{3} & 3 \\ -2 & \mathbf{1} & 1 \end{vmatrix} = -\frac{10}{7,}$$

$$x_3 = \frac{|\mathbf{B}_3|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} 2 & 0 & \mathbf{-3} \\ 4 & -5 & \mathbf{3} \\ -2 & 1 & \mathbf{1} \end{vmatrix} = \frac{1}{7}.$$

$$\text{So, } \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \frac{1}{7}\begin{bmatrix} -8 \\ -10 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 & -10 & 1 \end{bmatrix}^T.$$

Problem: #40 Use the cofactor method of this section to find the inverse ($\mathbf{A}^{-1}$) of:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -3 \\ 2 & -3 & -1 \\ -5 & 0 & -3 \end{bmatrix}.$$

Recall: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$.

$\det \mathbf{A} = 107.$ (trust me)

Cofactors: $A_{mn} = \text{(Checkerboard)(Fish)}$

$$A_{11} = (+)(9-0) = 9, \quad A_{12} = (-)(-6-5) = 11, \quad A_{13} = (+)(0-15)\ldots$$

$$[A_{mn}] = \begin{bmatrix} 9 & 11 & -15 \\ 12 & -21 & -20 \\ -13 & -4 & -14 \end{bmatrix}$$, and the transpose: $[A_{mn}]^T$ is…

$$[A_{mn}]^T = \begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$$, so

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T = \frac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}.$$

Problem: #59 Show that: $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$, and $\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 1$

Top row:

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(2 \cdot 2 - 1 \cdot 1) - 1(1 \cdot 2 - 1 \cdot 0) + 0 = 6 - 2 = 4. \text{ OR...}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \overset{c_3+(-2c_2)}{=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = 0 + (-)[2 \cdot (-3) - (-2 \cdot 1)] + 0 = 4, \text{ (where I use the last row).}$$

For the second determinant:

$$\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} \overset{c_3+(-2c_2)}{=} \begin{vmatrix} 2 & 1 & -2 & 9 \\ 1 & 2 & -2 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= +1 \begin{vmatrix} 2 & -2 & 9 \\ 1 & -2 & 3 \\ 1 & -3 & 1 \end{vmatrix} \overset{c_2+3c_1}{=} \begin{vmatrix} 2 & 4 & 9 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\overset{c_3+(-c_1)}{=} \begin{vmatrix} 2 & 4 & 7 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 7 \\ 1 & 2 \end{vmatrix} = 8 - 7 = 1.$$

Or just powering through on the first row...

$$\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 2A_{11} + A_{12} + 0 \cdot A_{13} + 9A_{14} = 2\begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} - 9\begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 2(1 \cdot (2 - 3) - 2(2 - 6)) - (-2)(1 - 3) - 9(1 \cdot (4 - 1) - 1 \cdot (4 - 2))$$

$$= 2(7) - 4 - 9(1) = 1.$$