MATH 2243: **Linear Algebra** & **Differential Equations**

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2.4: The Euler Method

Unsolvable DEQs: $\frac{dy}{dx} = f(x,y)$, with initial condition $y(x_0) = y_0$.

What if our methods for solving DEQs fail us? (And they often do!)

How do we use our knowledge of slope fields to generate solution curves?

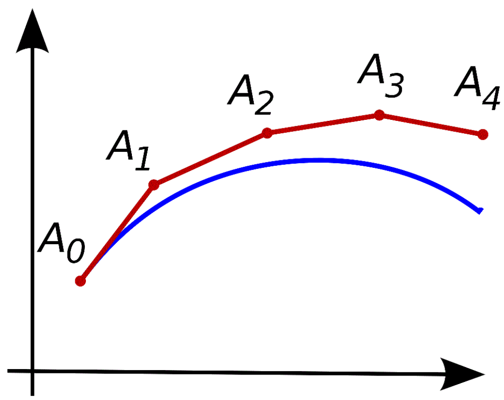
Definition: an **elementary function** is a function that is made up of a finite number of arithmetic operations ($+ - \times \div$), constants, exponentials, logarithms, or trigonometric functions. In other words, elementary functions are **most everything you've ever seen**.

The methods for solving DEQs we've learned so far result in elementary functions, but what if the solution isn't elementary?

Example: The anti-derivative of $y' = e^{-x^2}$ is a nonelementary function, so you can't easily solve for it.

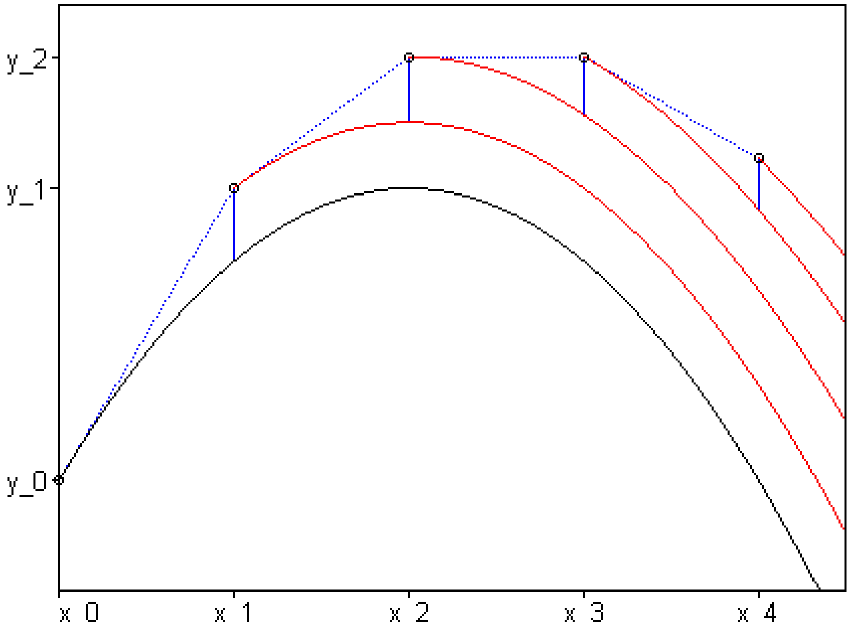
Our previous techniques cannot discover a solution for y', and it turns out that the solution is a nonelementary function (nonelementary examples: $f(x) = |x|$, $f(x) = \int_0^x e^{-t^2} dt$, $f(x) = \int_0^x \ln(\ln x) dt$, $f(x) = \int_0^x \sin(x^2) dt$, etc.).

The Euler Method: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$, $n \geq 0$, with step size h.



Local and Cumulative Errors

Local error is the amount to which the tangent line departs from the solution curve between each successive (x_n, y_n) and (x_{n+1}, y_{n+1}).



If you add all of these individual errors up along the way (adding the lengths of the blue lines in the example above), the total is the **cumulative error**.

Minimize error: shorten step size until desired accuracy is achieved.
Limitations:

- Each time you divide the step size in half, you double the number of calculations, and thereby lengthen the time it takes the computer to calculate. Desired accuracy may take thousands of years!
- Computers round off each calculation to some decimal point. The more calculations, the more error. Precious accuracy is slowly lost, and with small enough step size you may actually get less accuracy instead of more accuracy!

Which step size best for Euler Method accuracy?

Complexities involved in answering this require a separate course!
However, graphing results of successive step sizes combined with intuition regarding your function is often good enough.

What if you absolutely need more accuracy?

Improved methods are presented in subsequent sections of the book.

Also: If your unknown function y has a singularity, or is undefined in the domain you are analyzing (for example: near $x = 0$), Euler method will produce unreliable results there.

Example: Solution was singularity:

$y' = \frac{1}{x^2}$ with $y(-1) = 1$, what is $y(0)$ with step size $h = 0.2$?

For the simple equation, we can solve: $y = -\frac{1}{x} + C \quad \Rightarrow \quad 1 = -\frac{1}{-1} + C \quad \Rightarrow \quad C = 0.$

$\Rightarrow \quad y = -\frac{1}{x}$. So $y(0)$ is undefined, and $y \to \infty$ as $x \to 0$.

With Euler,

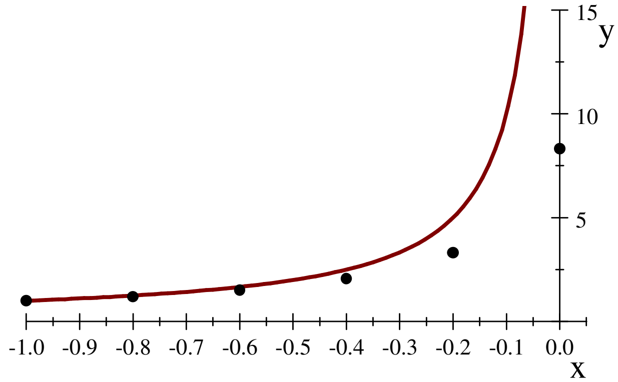
$$y_1 = y_0 + h \cdot f(x_0) = 1 + 0.2\left(\frac{1}{(-1)^2}\right) = 1.2$$

$$y_2 = y_1 + h \cdot f(x_1) = 1.2 + 0.2\left(\frac{1}{(-0.8)^2}\right) = 1.5125$$

$$y_3 = 1.5125 + 0.2\left(\frac{1}{(-0.6)^2}\right) = 2.0681$$

$$y_4 = 2.0681 + 0.2\left(\frac{1}{(-0.4)^2}\right) = 3.3181$$

$$y_5 = 3.3181 + 0.2\left(\frac{1}{(-0.2)^2}\right) = 8.3181 \stackrel{?}{=} \infty \qquad ?!?!$$



Exercises 

Problem: #8

Consider the initial value problem: $y' = e^{-y}, \quad y(0) = 0,$

with exact solution: $y(x) = \ln(x+1)$.

Apply Euler's method twice to approximate to this solution on the interval $\left[0, \frac{1}{2}\right]$;

first with step size $h_1 = 0.25$, then with step size $h_2 = 0.1$.

Compare the values of the approximations at $x = \frac{1}{2}$

with the value $y\left(\frac{1}{2}\right)$ of the actual solution ($y(\frac{1}{2}) = \ln\left(\frac{1}{2} + 1\right) \approx 0.405$).

Recall: Iterative formula: $y_{n+1} = y_n + h \cdot e^{-y_n}$.

$y_1 = 0 + (0.25)e^{-0} = 0.25$

$y_2 = 0.25 + (0.25)e^{-0.25} \approx 0.4447.$

$y_1 = 0 + (0.1)e^{-0} = 0.1$

$y_2 = 0.1 + (0.1)e^{-0.1} = 0.19048$

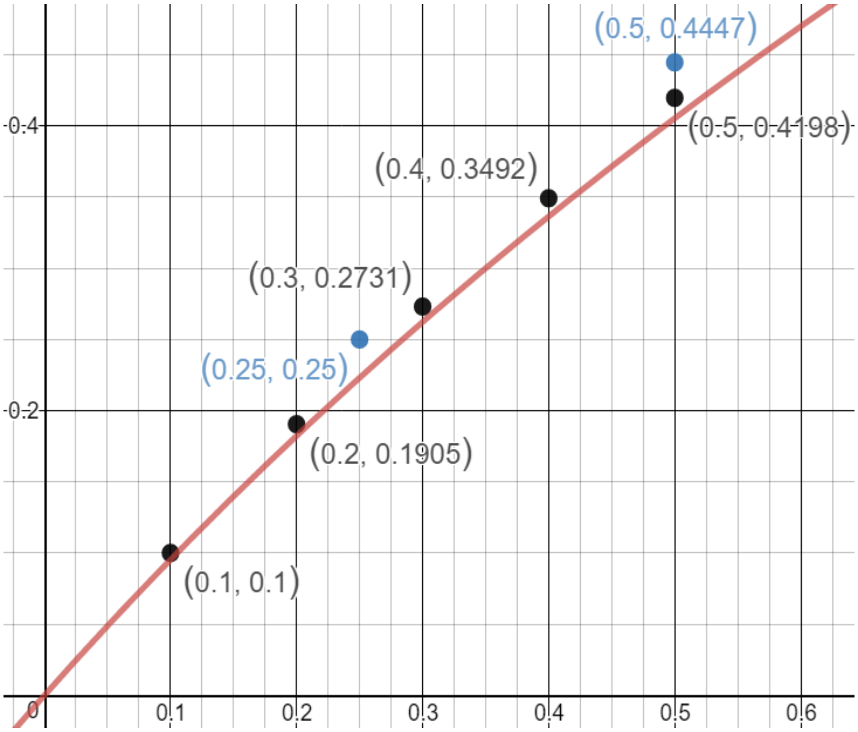
$y_3 = 0.19048 + (0.1)e^{-0.19048} = 0.27314$

$y_4 = 0.27314 + (0.1)e^{-0.27314} = 0.34924$

$y_5 = 0.34924 + (0.1)e^{-0.34924} = 0.41976$

Approximate values: $h_1 : 0.445$ and $h_2 : 0.420$;

True value? $\quad y\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2} + 1\right) \approx 0.405.$



Problem: **#10**

$y' = 2xy^2,\ y(0) = 1;$ Solution: $y(x) = \frac{1}{1-x^2}$.

First with step size $h_1 = 0.25$, then with step size $h_2 = 0.1$.

Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$

$y_1 = 1 + (0.25)(2 \cdot 0 \cdot 1^2) = 1,$

$y_2 = 1 + (0.25)(2 \cdot (0.25) \cdot 1^2) = 1.125.$

$y_1 = 1 + (0.1)(2 \cdot (0) \cdot 1^2) = 1,$

$y_2 = 1 + (0.1)(2 \cdot (0.1) \cdot 1^2) = 1.02,$

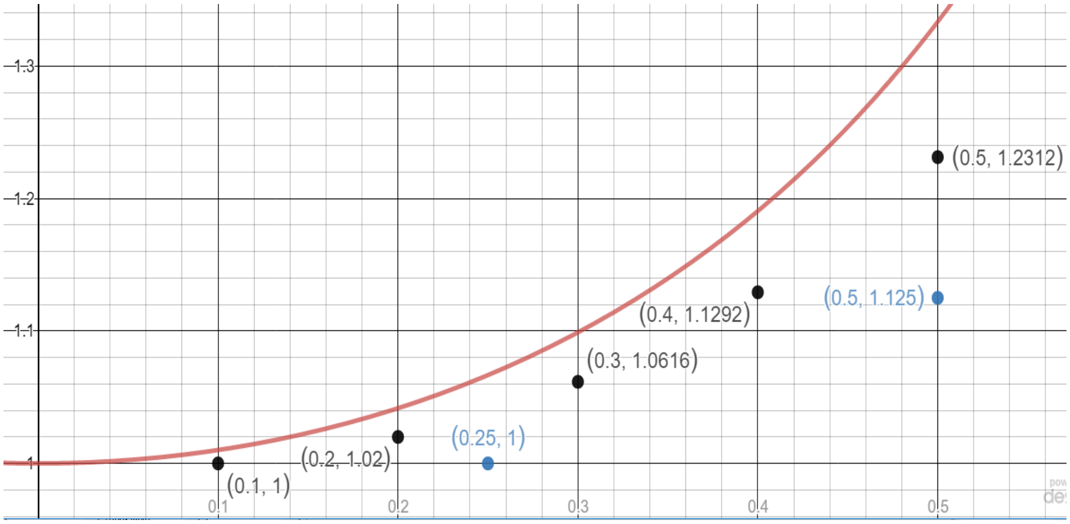
$y_3 = (1.02) + (0.1)\left(2 \cdot (0.2) \cdot (1.02)^2\right) = 1.0616,$

$y_4 = (1.0616) + (0.1)\left(2 \cdot (0.3) \cdot (1.0616)^2\right) = 1.1292,$

$y_5 = (1.1292) + (0.1)\left(2 \cdot (0.4) \cdot (1.1292)^2\right) = 1.2312.$

Approximate values: $h_1 = 1.125$ and $h_2 = 1.231$;

True value: $y\left(\frac{1}{2}\right) \approx 1.333$.



Note that for $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$, if we were to use the Euler method with step sizes $h = 0.1,\ 0.02$, and 0.005 for x in the interval $[0, 1]$; the following table would result…

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.005$
0.1	1.1000	1.1088	1.1108
0.2	1.2220	1.2458	1.2512
0.3	1.3753	1.4243	1.4357
0.4	1.5735	1.6658	1.6882
0.5	1.8371	2.0074	2.0512
0.6	2.1995	2.5201	2.6104
0.7	2.7193	3.3612	3.5706
0.8	3.5078	4.9601	5.5763
0.9	4.8023	9.0000	12.2061
1.0	7.1895	30.9167	1502.2090