MATH 2243: Linear Algebra & Differential Equations

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2.2: Equilibrium Solutions and Stability

If we can't solve a DEQ, then besides its slope field, what else can we figure out?

First, let's assume you are dealing with an autonomous DEQ.

(because even if you aren't, there's a trick you can do to convert your DEQ into one)

Autonomous DEQ: A DEQ which does not *explicitly* depend on t, for example: $\frac{dy}{dt} = f(y) = y(t) + a$ or $\frac{dy}{dt} = f(y,t) = y(t)^2 - by(t) + y(t)$. As opposed to DEQs which *do* depend explicitly on t, for example: $\frac{dy}{dt} = f(y,t) = y(t) + at^2 + b$.

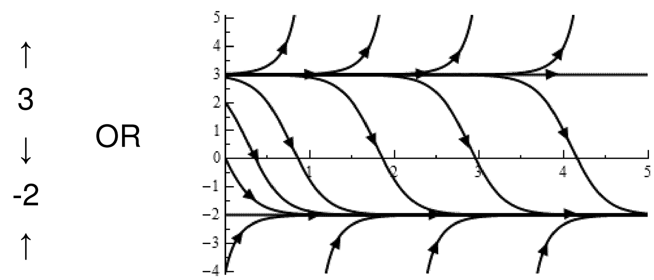
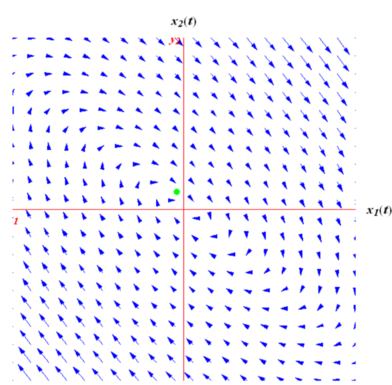


Figure 1: 1D Autonomous $y' = f(y) = y^2 - y - 6$



2D Nonautonomous (animated in class)

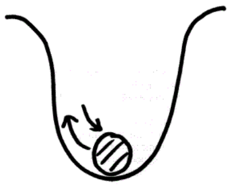
Critical points (of an autonomous DEQ $y' = f(y)$): These are solutions to the equation $f(y) = 0$ (i.e., y_0 such that $f(y_0) = 0$). These points are very important, because they divide up our domain into parts with qualitatively different behaviors, and provide us with important results, especially when exact solutions to the DEQ cannot be found. Examples of critical points in Fig. 1 above are the values $y_0 = -2$ and $y_0 = 3$.

Equilibrium Solution: If x_0 is a critical point, then the DEQ has the constant solution $x(t) = x_0$, called an equilibrium solution. Examples of equilibrium solutions in Fig. 1 above are the functions $y(t) = -2$ and $y(t) = 3$.

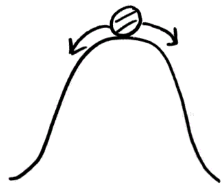
Stable/Unstable Critical Points:

A critical point $y = c$ of an autonomous first order DEQ is said to be **stable** provided that, for all initial values y_0 of the solution $y(t)$ sufficiently close to c, the solution $y(t)$ remains close to c, for all $t > 0$ (recall that $y_0 := y(0)$).

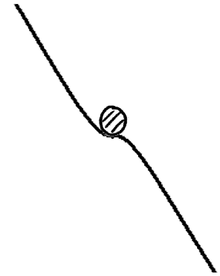
Otherwise, the critical point c is considered to be **unstable**. In other words, unstable means that no matter how close to c your initial value y_0 is, the solution $y(t)$ may drift away from c, for some $t > 0$.



Stable Equilibrium



Unstable Equilibrium



Unstable (Semi-Stable

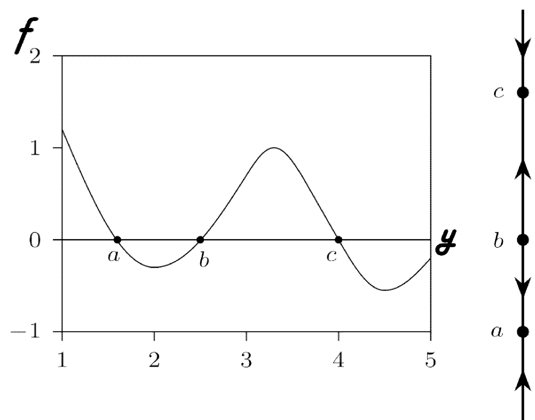
Phase Diagram: Is a type of diagram used to distinguish between qualitatively different parts of the domain. These different parts are separated by equilibria. Using the example in Fig. 1, the phase diagram would be: $\rightarrow \ -2 \ \leftarrow \ 3 \ \rightarrow$

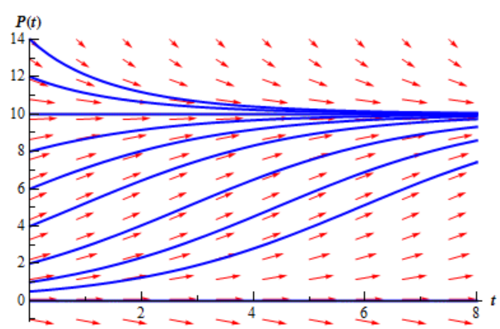
How to construct this?

- Find critical points
- Test points around critical points to see if $f > 0$ or $f < 0$.

Other Examples: funnels/spouts/sources/sinks/etc.

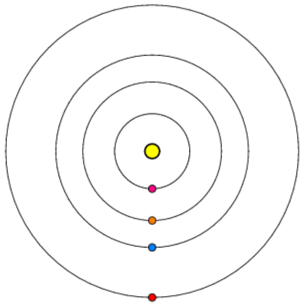
1D $\rightarrow -2 \leftarrow 3 \rightarrow 5 \rightarrow$



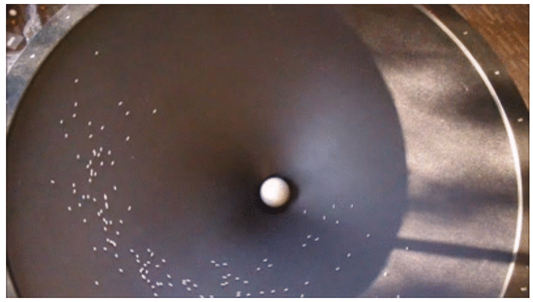


Logistic example of funnel

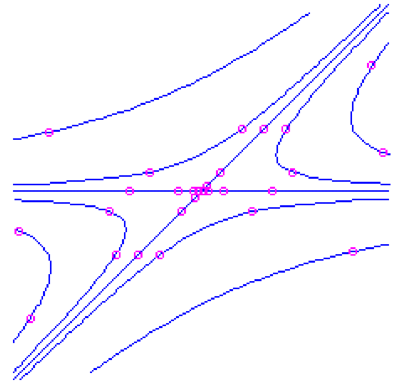
2D



Stable critical point (animated in class)



Stable critical point (animated in class)



Unstable critical point (animated in class)

More formally, a critical point c is **stable** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$|y_0 - c| < \delta$ implies that $|y(t) - c| < \varepsilon$, for all $t > 0$.

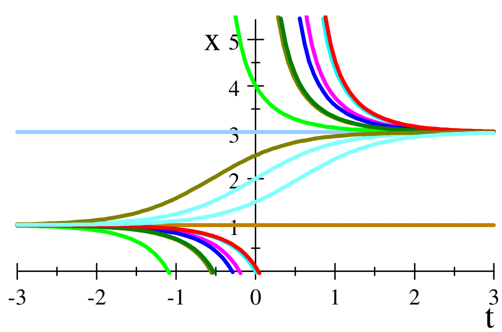
The critical point $y = c$ is **unstable** if it is not stable.

Bifurcation Point: Given a DEQ with parameter h (for example, $\frac{dx}{dt} = x(4-x) - h$), a bifurcation point is a point $h = h_0$ where the number of equilibrium points changes depending upon whether h is greater or lesser than h_0.

For $x' = x(4-x) - h$, note that when $h = 3$:

$$x(4-x) - 3 = 0$$

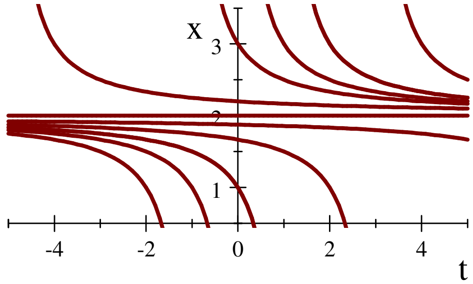
$$\Rightarrow \quad x \in \{3, 1\}. \text{ Two equilibrium points.}$$



$\frac{3(C-1)-1(C-3)e^{-2t}}{(C-1)-(C-3)e^{-2t}}$, for various C

However, when $h = 4$, we have: $x(4-x) - 4 = 0$

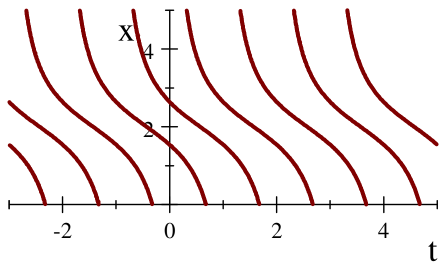
$$\Rightarrow \quad x^2 - 4x + 4 = (x-2)^2 = 0 \text{ when } x = 2. \text{ Only one critical point!}$$



$\frac{-2C+2t+1}{t-C}$, for various C

And when $h = 5$, we have: $x(4-x) - 5 = 0$

$\Rightarrow x \in \{2 \pm i\}$. So no (real) critical points.



$x = 2 - \tan\left(t + C - \frac{\pi}{2}\right)$, for various C

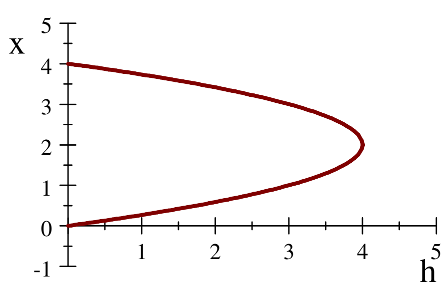
So more generally, solving $x(4-x) - h = 0$,

$$-x^2 + 4x - h = 0$$

$$\Rightarrow \quad x(h) = \frac{-4 \pm \sqrt{16 - 4(-1)(-h)}}{-2} = 2 \pm \sqrt{4-h}.$$

Two critical points when $h < 4$. Therefore, there is a bifurcation point at $h_0 = 4$.

Bifurcation Diagram: It's a diagram in the hx-plane (see above) which visualizes the bifurcation of the critical points ($f = 0$) as you vary a bifurcation parameter h.



Mapping critical points $x(h)$ in the hx-plane

Exercises

Problem: #10 **First solve $f(x) = 0$ to find the critical points of the autonomous DEQ $\frac{dx}{dt} = f(x) = 7x - x^2 - 10$. Then analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the DEQ. Next, solve the DEQ explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given DEQ, and verify visually the stability of each critical point.**

$7x - x^2 - 10 = (x-2)(x-5) = 0 \quad \Rightarrow \quad x \in \{2, 5\}$ are the critical points.

Then, looking at the sign of $f(x)$ on either sides of the critical points, we have: $f(1) = -4 < 0, \quad f(3) = 2 > 0, \quad f(6) = -4 < 0.$

This gives us the phase diagram: $\leftarrow \;\; 2 \;\; \rightarrow \;\; 5 \;\; \leftarrow$

Stable critical point: $x = 5$.

Unstable critical point: $x = 2$.

Funnel: Along the equilibrium solution $x(t) = 5$.

Spout: Along the equilibrium solution $x(t) = 2$.

Solution: If $x_0 \notin \{2, 5\}$, then :

$\frac{dx}{dt} = 7x - x^2 - 10 \quad \Rightarrow \quad \frac{1}{(x-2)(x-5)} dx = -dt$

Partial fractions:

$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$ when $1 = A(x-5) + B(x-2) = (A+B)x - (5A + 2B)$

$\Rightarrow \quad A + B = 0$ and $-5A - 2B = 1$

$\Rightarrow \quad B = -A, \quad -5A - 2(-A) = 1, \quad A = -\frac{1}{3}$ and $B = \frac{1}{3}$.

Therefore, $\frac{1}{(x-2)(x-5)} = \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5}$

Continuing our integration: $\int \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5} dx = -\int dt$ or $\int \frac{1}{x-5} - \frac{1}{x-2} dx = -3\int dt$

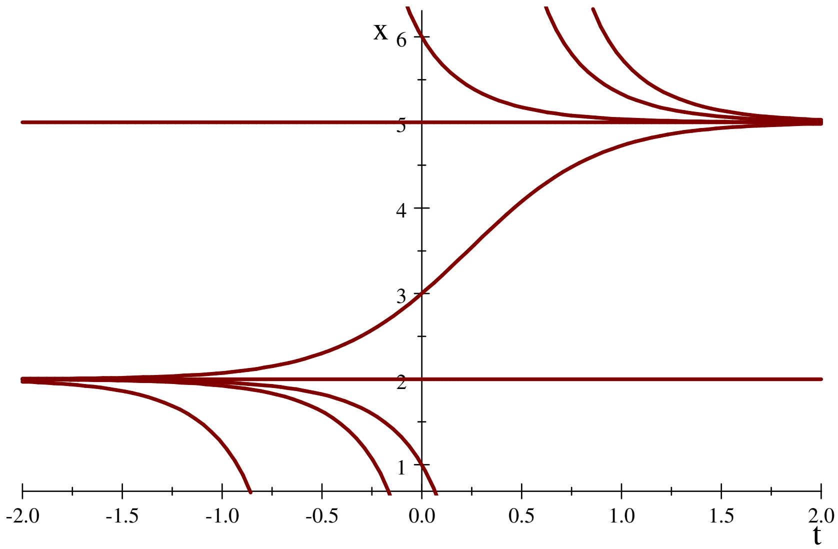
$\ln|x-5| - \ln|x-2| = -3t + C \quad \Rightarrow \quad \ln\left|\frac{x-5}{x-2}\right| = -3t + C$

$\frac{x-5}{x-2} = \pm e^C e^{-3t}$, and for any initial condition we have $\frac{x_0-5}{x_0-2} = \pm e^C$, so our solution becomes $\frac{x-5}{x-2} = \frac{x_0-5}{x_0-2}e^{-3t}$.

Solving explicitly for x: $\quad x - 5 = \frac{x_0-5}{x_0-2}e^{-3t}(x-2)$

$$x - \frac{x_0-5}{x_0-2}e^{-3t}x = -2\frac{x_0-5}{x_0-2}e^{-3t} + 5$$

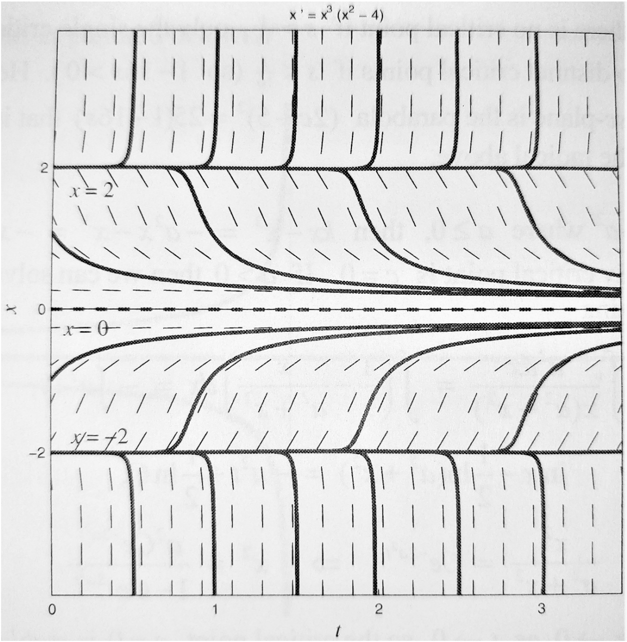
$$x = \frac{-2\frac{x_0-5}{x_0-2}e^{-3t}+5}{\left(1-\frac{x_0-5}{x_0-2}e^{-3t}\right)} = \frac{5(x_0-2)-2(x_0-5)e^{-3t}}{(x_0-2)-(x_0-5)e^{-3t}} = \frac{5(x_0-2)e^{3t}-2(x_0-5)}{(x_0-2)e^{3t}-(x_0-5)}.$$



$$\frac{5(x_0-2)e^{3t}-2(x_0-5)}{(x_0-2)e^{3t}-(x_0-5)}$$

Problem: #18 **Use a computer system or graphing calculator to plot a slope field and/or enough solution curves to indicate the stability or instability of each critical point of** $\frac{dx}{dt} = x^3(x^2-4)$. **(Some of these critical points may be semi-stable in the sense mentioned in the Example 6**).

$f(x) = x^3(x^2-4) = 0$ when $x \in \{0, \pm 2\}$.



The critical points $x = 2$ and $x = -2$ are unstable, while the critical point $x = 0$ is stable.