Instructor: Jodin Morey moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

1.4: Separable Equations and Applications

Separation of Variables:

How do we solve a differential equation when y' is written not only in terms of x, but also in terms of y like: $y' = f(x,y)$.

Can't just integrate right away, but can we multiply both sides of equation by some factor which allows us to then integrate? Sometimes! When we can, this something is called an **integrating factor**. It's called this because after multiplying, both sides of the equation take the form of a derivative, allowing us to integrate.

Example: $\frac{dy}{dx} = \frac{9x}{4y}$

$4ydy = 9xdx$ or $4y\frac{dy}{dx} = 9x$ (integrating factor $4y$)

$\int 4ydy = \int 9xdx \Rightarrow 2y^2 = \frac{9}{2}x^2 + c$ (implicit solution)

$y = \pm\sqrt{\frac{9}{4}x^2 + \frac{c}{2}}$.

Example: $\frac{dy}{dx} = 2x(y+2)$

$\frac{1}{y+2}\frac{dy}{dx} = 2x$ when ??

$y(x) \neq -2$.

$\int \frac{1}{y+2}dy = \int 2xdx$

$\ln|y+2| = x^2 + c$,

$e^{\ln|y+2|} = e^{x^2+c}$

$y + 2 = \pm e^{x^2}e^c$

$y + 2 = de^{x^2}$, where $d \neq 0$.

So our (partial) solution is: $y = de^{x^2} - 2$, where $d \neq 0$.

But what happens if $y(x) = -2$?

Observe that for both sides of the given differential equation, we have:

$$\frac{dy}{dx} = \frac{d(-2)}{dx} = 0 \quad \text{and} \quad 2x(y+2) = 2x(-2+2) = 0.$$

So, $y(x) = -2$ IS a solution. And the FULL solution is: $y = de^{x^2} - 2$, for any $d \in 5$ (d can be zero!).

More Formally

Separable equations: If you can (re)write $\frac{dy}{dx} = f(x,y)$ as $\frac{dy}{dx} = g(x)h(y)$ (an expression in x multiplied by an expression in y), with $h(y) \neq 0$, then rearrange the equation as: $\frac{1}{h(y)}dy = g(x)dx$.

Now you hope to integrate both sides of the equation:

$$\int \frac{1}{h(y)}dy = \int g(x)dx.$$

So that, $F(y(x)) = G(x) + C$, where $F(y(x)) = \int \frac{1}{h(y)}dy$ and $G(x) = \int g(x)dx$.

Implicit Solutions:

Given $\frac{dy}{dx} = f(x)$, with initial condition $y(0) = 2$.

Usually we want to find an explicit solution y of the form: $y(x) =$ (some expression in x).

However, for $x + yy' = 0$ we solve to find an implicit solution:

$$x^2 + y^2 - 4 = 0 \quad \Rightarrow \quad y(x) = \pm\sqrt{4 - x^2} \qquad \text{(which is it: + or − ??)}$$

Only one of these satisfies the given initial condition: $y(0) = 2$.

So in this case, the solution actually is: $y(x) = \sqrt{4 - x^2}$.

Moral of Story: Not every explicit solution obtained from an implicit solution satisfies all initial conditions (so check them!).

Definition: An equation $K(x,y) = 0$ is called an implicit solution of a DEQ if it is satisfied by some solution y of the DEQ.

By this definition $(y - 2x)(x^2 + y^2 - 4) = 0$, where we have multiplied both sides of the equation by $y - 2x$ is still an implicit solution.
However, this new equation also tells us $y = 2x$, which does not solve the DEQ.

These types of erroneous conclusions can enter our calculations when we use integrating factors while solving a DEQ.

So we must check the solutions we obtain.

We also saw in a previous section that multiplying by an integrating factor can cause us to lose singular solutions. So we must keep track of the different "cases" where we might be dividing by zero.

Exponential Growth/Decay:

Equations of the form: $\frac{dy}{dx} = ky$ have solutions...

$y = Ce^{kx}$, where $C \in \mathbb{R}$. How do we know?

$\int \frac{1}{y} dy = \int k dx$, when $y \neq 0$,

$\Rightarrow \quad \ln|y| = kx + c \quad \Rightarrow \quad |y| = e^{kx+c}$

$\Rightarrow \quad y = Ce^{kx}$, where $C = \pm e^{c} \in \mathbb{R} \backslash \{0\}$ (any real number except zero)

When $y = 0$, note that this is a solution to our differential equation. Additionally, we can incorporate $y = 0$ as a solution into our previous solution ($y = Ce^{kx}$) if we let $C = 0$. So, we edit our conditions on C so it can be any real number, or with fancy notation: $C \in \mathbb{R}$.

Examples:

Population Growth: $P'(t) = kP$, where $k = \beta - \delta$ and β, δ are the birth and death rates, respectively.

Radioactive Decay: $N'(t) = -kN$, where N is the number of atoms, and $k > 0$ is the rate at which N decays.

Drug Elimination: $A'(t) = -\lambda A$, where $\lambda > 0$ is the "elimination constant" for that drug.

Underwater Light Intensity: $\frac{dI}{dx} = -kI$, where I is the amount of light, x is the depth below the surface, and $k > 0$ depends upon the properties of the water.

Newton's law of cooling

The rate of change (with respect to time) of the Temperature $T(t)$ of an object immersed in an ambient temperature A is proportional to the difference $A - T$. In other words: $\frac{dT}{dt} = k(A - T)$, where k is a constant that depends upon the heat conductivity of the surrounding medium (air?).

Example: A cake is removed from a 210°F oven, and left to cool in room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F?

$\frac{dT}{dt} = k(A - T) = k(70 - T)$

$\frac{1}{70-T}dT = kdt$, when $T \neq 70$. (what happens when $T = 70$?)

$\int \frac{1}{70-T}dT = \int kdt \qquad -\ln|70 - T| = kt + c$

$kt = -\ln|70 - T| - c$

$e^{kt} = e^{-\ln|70-T|-c} = e^{-\ln|70-T|}e^{-c} = \frac{1}{70-T}C$, where $C \neq 0$. Then ??

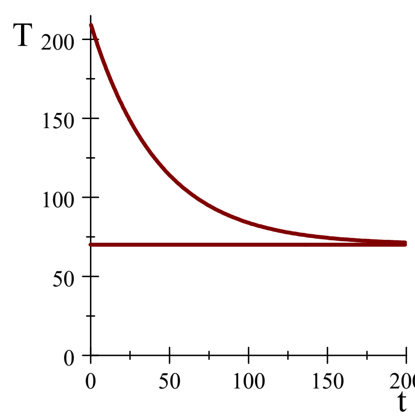
$e^{k\cdot 0} = \frac{1}{70-210}C \qquad \Rightarrow \qquad -140 = C$. Then ??

$e^{k\cdot 30} = \frac{1}{70-140}(-140) = 2$

$\Rightarrow \qquad 30k = \ln 2, \qquad k = \frac{\ln 2}{30}$. Then ??

$e^{\frac{\ln 2}{30}\cdot t} = \frac{1}{70-100}(-140) = \frac{14}{3}$

$\frac{\ln 2}{30}t = \ln\frac{14}{3}, \qquad t = \frac{30\ln\frac{14}{3}}{\ln 2} \approx 67$ minutes till the cake will be 100°F.



$e^{\frac{\ln 2}{30}t} = \frac{-140}{70-T}$

Exercises

Problem: #26 Find explicit particular solutions of the initial value problem:

$\frac{dy}{dx} = 2xy^2 + 3x^2y^2$, with init. cond. $y(1) = -1$.

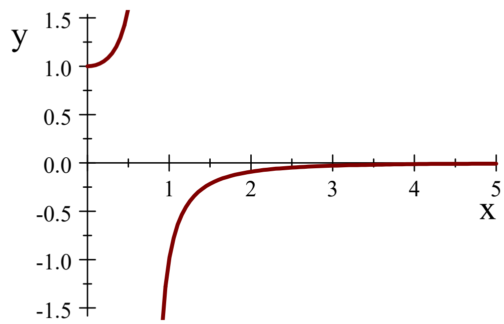
$\frac{dy}{dx} = y^2(2x + 3x^2)$

$\frac{1}{y^2}\frac{dy}{dx} = 2x + 3x^2 \qquad \Rightarrow \qquad \int \frac{d}{dx}\left(-\frac{1}{y}\right)dx = \int (2x + 3x^2)dx \qquad$ (integrating factor $\frac{1}{y^2}$)

$\Rightarrow -\frac{1}{y} = x^2 + x^3 + C, \qquad y(x) = -\frac{1}{x^2+x^3+C}. \qquad$ Are we done?

$-1 = -\frac{1}{1^2+1^3+C}, \qquad 2 + C = 1, \qquad C = -1.$

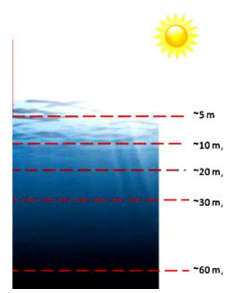
So, $y(x) = -\frac{1}{x^2+x^3-1}$.



$-\frac{1}{x^2+x^3-1}$

Underwater Light Intensity

Problem: **#45*** **Natural (Exponential) Growth and Decay**: $\frac{dy}{dx} = ky$, where k is a constant.



The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation: $\frac{dI}{dx} = -1.4I$.

Part a. **At what depth is the light intensity** I **half of the surface light intensity** $I_0 := I(0)$?

$\frac{1}{I}dI = -1.4dx$, for $I \neq 0$. $\qquad$ (integrating factor $\frac{1}{I}$)

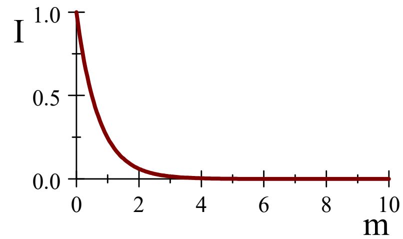
$\int \frac{d}{dx}\ln|I|dx = -1.4\int dx$

$\ln|I| = -1.4x + C$

$e^{\ln|I|} = e^C e^{-1.4x}$

The light intensity at a depth of x meters is therefore given by $I(x) = I_0 e^{-1.4x}$.

(how do I know $e^C = I_0$?)   Are we done?



$e^{-1.4x}$

Recall: "At what depth is the light intensity I, half of the surface light intensity, I_0 (where $x = 0$)?"

We solve the equation $\frac{1}{2} I_0 = I_0 e^{-1.4x}$.

$e^{-1.4x} = \frac{1}{2}$,

$-1.4x = \ln\frac{1}{2}$, $\Rightarrow$ $x = \frac{-\ln 2}{-1.4} \approx 0.495\ m$ below the surface.

Part b.   **What is the intensity at a depth of** $10\ m$ **(in terms of** I_0**)?** $I(x) = I_0 e^{-1.4x}$

At depth 10 meters, the intensity is: $I(10) = I_0 e^{-1.4 \cdot 10} \approx (8.32 \times 10^{-7}) I_0$.   (very dark)

Population Growth

Problem: For a population of kangaroos, assume that 10% of kangaroos become pregnant every year and have, on average, two joeys (babies) per litter. Also assume that 10% of kangaroos die annually.

a) **Given an exponential growth model, what is the DEQ which describes the population**?

$\beta = 0.2$, $\delta = 0.1$.

So $k = \beta - \delta = 0.1$,

and $P'(t) = kP = 0.1P$.

b) **What is the general solution**?

Solving: $\frac{1}{P}\frac{dP}{dt} = 0.1$ (assuming $P(t) \equiv 0$, which is a solution!)

$\Rightarrow$ $\int \frac{d}{dt} \ln P\,dt = \int 0.1\,dt$

$\Rightarrow$ $\ln P = 0.1t + c \Rightarrow P = Ce^{0.1t}$, where $C > 0$.

Note that since $P(t) \equiv 0$ is a singular solution, we can generate the general solution as $P = Ce^{0.1t}$, where $C \geq 0$.

c) If you know that in the 2011 there were 34 million kangaroos, how many kangaroos are predicted by this model this year?

Let t represent the number of years since 2011. So we have the initial condition $P(0) = 34$.

$34 = Ce^0 \quad \Rightarrow \quad P = 34e^{0.1t}$

$P(10) = 34e^{0.1(10)} \approx 92.422$ kangaroos!