MATH 2243: Linear Algebra & Differential Equations

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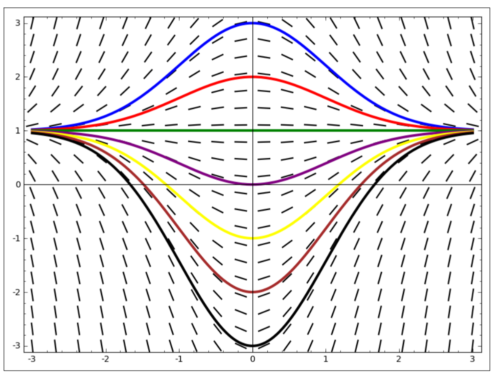
1.3: Slope Fields and Solution Curves

How to solve: $\frac{dy}{dx} = f(x,y)$? Like this? $\int \frac{dy}{dx} dx = \int f(x,y) dx$?!? (what does this even mean?)

$f(x,y)$ may involve the unknown function $y(x)$, and therefore integrating $f(x,y)$ with respect to x may not be possible in this way. We need another method!

While not all differential equations are **analytically** solvable (we can't solve them exactly), we can nonetheless draw a **slope field** for equations $y' = f(x,y)$. Do this by choosing any point (x,y), plug these values into $f(x,y)$ and this gives you a slope (a number). Then, graph a short line at (x,y) having the slope $y' = f(x,y)$. Repeat as needed. See the short slope marks in the graph below, and some particular solutions drawn through them.



Drawing paths in the plane that are parallel to the nearby slope marks (as in the graph above) gives you a **solution curve**, a curve representing a solution to your DEQ.

WolframAlpha.com

Type your DEQs into wolfram (for example if $y' = f(x,y)$) using syntax like:

"slope field f(x,y)".

For $\frac{dy}{dx} = x^2 - 2$, this would be "slope field $x^2 - 2$".

Existence and Uniqueness of Solutions:

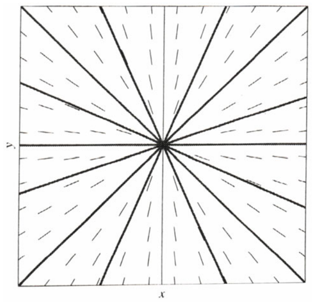
No one wants to search for something that doesn't **exist**!

Example: $y' = \frac{1}{x}$, $y(0) = 0$.

$\int \frac{1}{x} dx = \ln|x| + c.$

Not defined for initial condition $y(0) = 0$, (since $\ln x$ isn't defined when $x = 0$).

If we do have a solution, we'd also like to be sure there aren't any other solutions going through our initial conditions. That is, we want the solution to be **unique**.



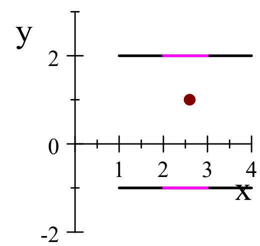
Sols to: $xy' = y$

Existence and Uniqueness Criteria:

Given: $\frac{dy}{dx} = f(x,y)$, and initial cond. $y(a) = b$.

Assume both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous on some rectangle R (containing (a,b) in its interior).

[For a concrete example, let's say f and $\frac{\partial}{\partial y}f$ are continuous in the rectangle $x \in (1,4)$ and $y \in (-1,2)$ where $(a,b)=\left(2.6,\ 1.0\right)$.]



Then, **Theorem 1.3** from the book guarantees the existence & uniqueness of a **local** solution in R, in some interval $x \in I$ (still containing a), but possibly smaller than the width of R.

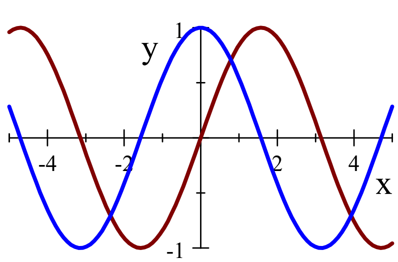
[Continuing our previous concrete example, the interval I might turn out to be $x \in (2,3)$, and observe $a = 2.6$ is still in this interval.]

Particularly, continuity of $f(x,y)$ guarantees existence on some I,

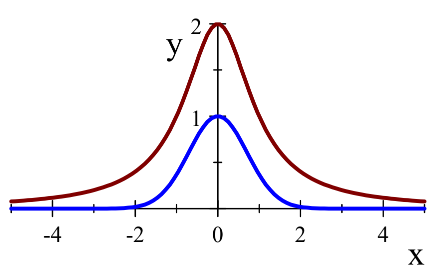
and continuity of $\frac{\partial}{\partial y}f(x,y)$ guarantees uniqueness of that solution.

In addition to being continuous, if $\frac{\partial}{\partial y}f(x,y)$ is also **bounded** for all x *and* y, then **global** existence/uniqueness (on $\mathbb{R}$) of a solution y is guaranteed.

But which functions are bounded?
Polynomials are not, but the following functions are: $\sin x$, $\cos x$, $\frac{2}{1+x^2}$, $e^{-|x|^2}$, etc.



$\sin x$, $\cos x$



$\frac{1}{1+x^2}$, $e^{-|x|^2}$

Another way to ensure that you have global existence (on $\mathbb{R}$) is to check if the DEQ is **linear first-order** (i.e. $ay' + by + c = d$). *ALL* such differential equations have global existence for their solutions.

What is it to be linear? What is it to be first order?

Negative results from the above tests **tell you nothing**. In other words, even if these tests fail, the solution may still have existence/uniqueness/globalness! (you may have to think a bit about this previous sentence.)

Example: $xy' = 2y$ with $y(-1) = 1$.

$$y' = \tfrac{2}{x}y$$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$, when $x, y \neq 0$.

(although observe the given DEQ is satisfied when $y \equiv 0$, or when $x = 0$ and $y(0) = 0$)

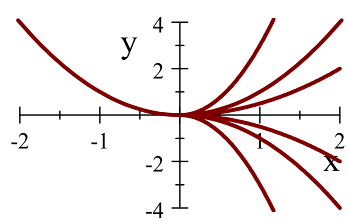
(also, the DEQ is never satisfied when $x = 0$ and $y(0) \neq 0$)

$$\int \tfrac{1}{y}\,dy = \int \tfrac{2}{x}\,dx \quad \Rightarrow \quad \ln|y| = 2\ln|x| + c = \ln|x^2| + c \quad \Rightarrow$$

$y = \pm e^c x^2 = C_1 x^2$, where $C_1 \in \mathbb{R}$, around $y(-1) = 1$.

Let's choose $C_1 = 1$. Observe that $y(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ C_2 x^2 & \text{if } x > 0, \end{cases}$

is a continuous solution for any $C_2 \in \mathbb{R}$.



Uniqueness is lost at $(0,0)$ where $f = \frac{2}{x}y$ and $\partial_y f = \frac{2}{x}$ fail to be continuous.

Exercises

Problem: #18 Determine whether existence of at least one solution of the initial value problem $y\frac{dy}{dx} = x - 1;\ y(1) = 0$ is guaranteed. If so, then is uniqueness of that solution also guaranteed?

$f(x,y) = \frac{x-1}{y}$, $\quad \frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ $\quad$ Continuous near $(1,0)$?

Neither $f(x,y) = \frac{x-1}{y}$ nor $\frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ is continuous near $(1,0)$, so the existence-uniqueness theorem guarantees nothing. There still may be existence/uniqueness of a solution, but these tools don't tell us either way.

Problem: #28 This problem will illustrate that if the hypotheses of **Theorem 1.3** above are NOT satisfied, then the initial value problem $y' = f(x,y),\ y(a) = b$ may have either

- no solutions (no existence)
- finitely many solutions, or (existence, possibly uniqueness)
- infinitely many solutions. (no uniqueness)

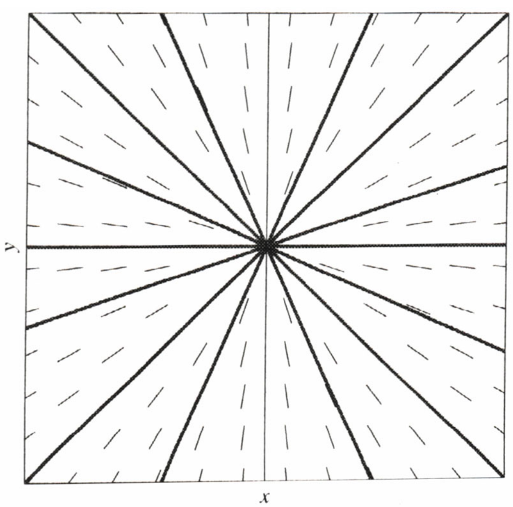
Verify that if k **is a constant, then the function** $y(x) = kx$ **satisfies DEQ**: $xy' = y$ **for all** x.

Taking the derivative of the solution, we have: $y' = (kx)' = k$.

Substituting this into our DEQ $xy' = x(kx)' = kx$, which is equal to $y(x) = kx$.

Note that $y' = \frac{y}{x}$ is not continuous at $(0,b)$. So Theorem 1.3 is NOT satisfied.

Construct a slope field, and several of the straight-line solution curves.



Then determine how many different solutions the initial value problem: $xy' = y,\ y(a) = b$ **has for various** (a,b). — **one, none, or infinitely many**.

Note that $f(x,y) = \frac{y}{x}$ is certainly continuous when $x \neq 0$, as is $\frac{\partial}{\partial y}f = \frac{1}{x}$.

So our theorem guarantees us a unique solution when $a \neq 0$, but tells us nothing when $a = 0$.

However, we visually verified above that there were infinitely many solutions when $a = 0, b = 0$.

Note that when $a = 0, b \neq 0$, that $xy' = y$ has no solutions, because we end up with $0y' = b \neq 0$.

So, the initial value problem has...

- a unique solution off the y-axis where $a \neq 0$;
- infinitely many solutions through the origin where $a = b = 0$;
- no solution if $a = 0$ but $b \neq 0$ (so if the point (a,b) lies on the positive or negative y-axis).

Problem: #20 Determine whether existence of at least one solution of the initial value problem $\frac{dy}{dx} = x^2 - y^2;\ y(0) = 1$ is guaranteed and, if so, whether uniqueness of that solution is guaranteed.

Both $f(x, y) = x^2 - y^2$ and $\frac{\partial f}{\partial y} = -2y$ are continuous near $(0, 1)$ (and everywhere since they are polynomials!), so the theorem guarantees both existence and uniqueness of a solution in some (actually any!) rectangle containing $x = 0$.

What about global existence?

$\frac{\partial f}{\partial y} = -2y$ is not bounded (goes to ∞ as $y \to \infty$), and $f(x, y)$ is not linear. So global existence is not guaranteed.