MATH 2243: **Linear Algebra** & **Differential Equations**

Instructor: Jodin Morey   moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

Book: Differential Equations & Linear Algebra 4th edition, Edwards & Penny

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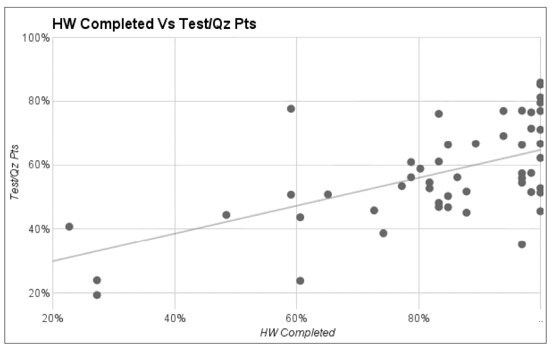
Syllabus

Introduction to the Course

Quizzes: 10 of them given by your discussion section instructor.

Grading: It is *extremely important* you do the homework, especially the problems toward the end of the homework assignment. Indeed, almost nothing is more important than the amount of effort you put into the homework. Don't fool yourself into thinking you can afford to skip an assignment.



Quizzes/Exams: If you ever have a question about your grade, you can always email your discussion section instructor, or visit them in office hours.

Fast Paced!

PRO TIP

♦ You will absorb much more information during lecture and discussion section if you have already oriented yourself to the concepts by **reading the sections of the textbook ahead of time**.

♦ Also, students have been shown to learn best when digesting smaller chunks of information separated by periods of sleep. Subconscious processes occur during sleep (and inbetween learning sessions), that put the new knowledge into our long-term memory. So, **spread your learning out over the week/semester**, **and get good sleep**.

PDFs: My PDFs (including this one) will be available as a PDF on the lecture Canvas.

Review Differentiation

Derivatives of...

♦ **Constant Functions**: If $y = c$, where c is some constant (for example, $y = 7$), then $y' = 0$.

♦ **Power Functions**: If $f(x) = x^r$, where r is a real number, then $\frac{d}{dx}(x^r) = rx^{r-1}$ (for example, $\frac{d}{dx}(x^5) = 5x^4$)

This includes if $f(x) = x$, where $\frac{d}{dx}(x^1) = 1 \cdot x^0 = 1$.

♦ **Polynomials**: Using Constant Multiple Rule and Sum/Difference Rule

$y = ax^2 + bx^3 - c$, where a, b, c are some constants

$y' = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx^3) - \frac{d}{dx}(c)$

$y' = a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x^3) - 0$

$y' = 2ax + 3bx^2$.

Trigonometric Differentiation

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

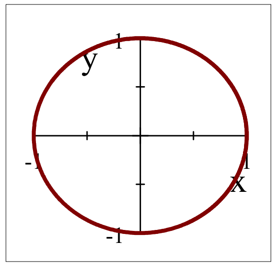
♦ **Product Rule**: If f and g are both differentiable, then $\frac{d}{dx}[f \cdot g] = g\frac{d}{dx}(f) + f\frac{d}{dx}(g)$ or $\frac{d}{dx}(f)g + f\frac{d}{dx}(g)$.

♦ **Quotient Rule**: If f and g are differentiable, then $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$.

♦ **The Chain Rule**: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = f'(g(x)) \cdot g'(x)$.

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

♦ **Power Rule Combined with the Chain Rule**: If n is any real number and $g(x)$ is differentiable (for example $g(x) = \cos x$), then $\frac{d}{dx}g^n = ng^{n-1}\frac{dg}{dx}$ (so $\frac{d}{dx}(\cos x)^5 = 5(\cos x)^4(-\sin x) = -5\sin x \cos^4 x$).



Implicit Differentiation

Some algebraic expressions cannot be expressed "**explicitly**," which means it cannot be expressed as a **function** of the form $y = f(x)$.

For example, the equation for a unit circle: $x^2 + y^2 = 1$. In this case, the closest we can get is to split it up into $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ (the upper and lower half of the unit circle). So how do you find the derivative of these "implicit" types of expressions?

Implicit Differentiation: First recall that when we take a derivative, we do so with respect to a particular variable. In the case of the circle above, the most obvious choice would be to take the derivative with respect to x. So we take the derivative $\frac{d}{dx}$ of both sides of the equation $x^2 + y^2 = 1$. When we do this, we keep in mind the **chain rule**, and consider any other variable than x we run across as a function of x. In other words, we think of y as $y(x)$. For example:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1),$$

$$(x^2)' + (y^2)' = 0,$$

$$2x(x)' + 2y(y)' = 0, \qquad \text{(using the chain rule)}$$

$$2x(1) + 2yy' = 0,$$

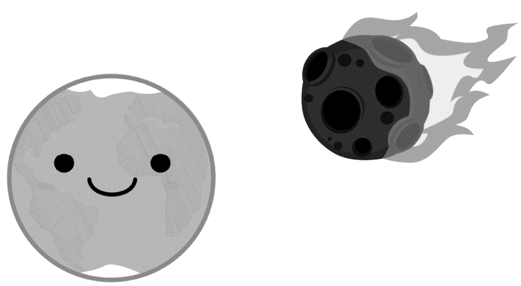
$$2yy' = -2x,$$

$$y' = -\frac{x}{y} \text{ or } \frac{dy}{dx} = -\frac{x}{y}.$$

So, plug in any point (a, b) from the unit circle, and the derivative $y' = -\frac{a}{b}$ tells you the slope of the circle at the point (a, b).

1.1: Differential Equations and Mathematical Models

Mathematical Modeling



Real-life situations we study often involve values changing over time (like location). How do we:

- Make predictions for these situations?
- Represent them mathematically?

Assume we can measure the rate of change of some phenomenon $y(t)$ (for example, location), and we measure it to be some constant k.

Recall rate of change (over time) is $\frac{d}{dt}$, so we can write: $\frac{d}{dt}(y(t)) = \frac{dy}{dt} = k$.

This is a (very simple) differential equation. More complicated situations require us to write down more complicated differential equations.

If we can manage to solve the differential equation (which means finding what $y(t)$ is), then we can predict the value of y (location) at time t.

Differential Equations

By now you have seen…

$y' = f(x)$ or $\frac{dy}{dx} = f(x)$

which implies (⇒): $\int \frac{dy}{dx} dx = \int f(x)dx \quad \Rightarrow \quad y = \int f(x)dx;$

Example: $\frac{dy}{dx} = x^2 + \pi \quad \Rightarrow \quad \int \frac{dy}{dx} dx = \int (x^2 + \pi)dx \quad \Rightarrow \quad y = \frac{1}{3}x^3 + \pi x + c.$

Exponential Growth/Decline:

Equations of the form: $y' = ky$, for some constant k.

Have solutions: $y = Ce^{kx}$. [Result 1]

Proof: If we differentiate: $\frac{dy}{dx} = kCe^{kx} \quad \Rightarrow \quad y' = ky.$ ■

Example: **Population** $P(t)$ 

Assume that yearly growth **rate** of a population of rabbits $P'(t)$ is proportional to its current population level, written: $\frac{dP(t)}{dt} = kP(t)$. Also, assume we know k to be 1.05. Can we determine the function $P(t)$, which will tell us the population at any point in time?

From Result 1 above we have:

$\frac{dP}{dt} = 1.05P \quad \Rightarrow \quad P = Ce^{1.05t}.$

Doing the math manually (which you will need to do on a test), we get:

$\frac{1}{P}dP = 1.05dt$, when $P(t) \neq 0$.

$\Rightarrow \quad \int \frac{1}{P}dP = \int 1.05dt \quad \Rightarrow \quad \ln|P| = 1.05t + c$

$\Rightarrow \quad P = e^{c+1.05t} = Ce^{1.05t}$, where $e^c = C > 0$. (why positive?)

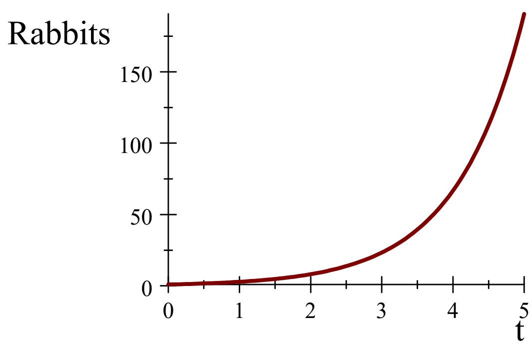
Note, if $P(t) = 0$, we have $\frac{dP}{dt} = 1.05P = 0$, and $P \equiv 0$ is a solution (since $\frac{d0}{dt} = 0$). So, $C \geq 0$.

What happens as $t \to +\infty$?

What if $k = -1.05$?

$P = Ce^{-1.05t}$

What happens as $t \to -\infty$ and $k = 1.05$?



$e^{1.05t}$

Second Order Differential Equations

Order of a DEQ is equal to the order of the highest derivative appearing in it.

The examples above are **separable 1st order DEQs** (from Calc 1/2).

We will also study **2nd order DEQs**.

Examples: $y'' = 9y, \quad y'' = -4y$.

We will learn very shortly how to discover solutions to these problems.

Today, let us just verify that $y = e^{\pm 3x}$ are solutions to $y'' = 9y$,

and that $y = \cos 2x, \ \sin 2x$ are solutions to $y'' = -4y$.

For solution: $y = e^{3x}$ of $y'' = 9y$, How should we proceed?

$y' = 3e^{3x}$ and $y'' = 9e^{3x} = 9y$.

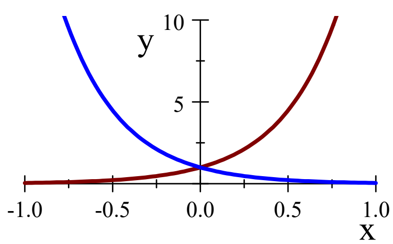
The verification for $y = e^{-3x}$ proceeds similarly.

For solution: $y = \cos 2x$ of $y'' = -4y$, How should we proceed?

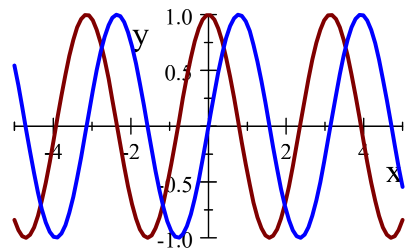
$y' = -2\sin 2x$ and $y'' = -4\cos 2x = -4y$.

The verification for $y = \sin 2x$ proceeds similarly.

Solutions look very different, even though differential equations were very similar!



e^{3x} and e^{-3x}



$\cos 2x$ and $\sin 2x$

Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - A)$ where T is the temperature of the object in question, and A is the temperature of the surrounding medium, for some $k > 0$.

Torricelli's Law: $\frac{dV}{dt} = -k\sqrt{y}$ where V is the volume of water in a draining tank,

y is the depth of the water, for some $k > 0$.