Orbital Mechanics Flashcards made to prepare for the Oral Exams

Lagrange equations of the First Kind	Treat constraints explicitly as extra equations, often using Lagrange multipliers $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{i=1}^{c} \lambda_i \frac{\partial f_i}{\partial q_i} = 0$, for each of c constraint equations f_i.
Lagrange equations of the First Kind Examples	Pendulum - Unconstrained: $\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$. Constraint: $x^2 + y^2 - \ell^2 = 0$. EOM: $m\ddot{x} = 2\lambda x$, $m\ddot{y} = 2\lambda y$, $x^2 + y^2 - \ell^2 = 0$. $\overline{L} = +\lambda(x^2 y^2 - \ell^2)$ $\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{x}_i} = \frac{\partial \overline{L}}{\partial x_i}$, $\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{y}_i} = \frac{\partial \overline{L}}{\partial y_i}$, $\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{\lambda}} = \frac{\partial \overline{L}}{\partial \lambda}$
Lagrange equations of the Second Kind	Incorporate the constraints directly by judicious choice of generalized coordinates.
Newton's laws Benefits/Draw-backs	**Benefits**: Can include non-conservative forces like friction **Draw-backs**: Must include constraint forces explicitly and are best suited to Cartesian coordinates
Hamiltonian system	$2n$ ODEs where H is smooth real valued defined on open set in $\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n$. Satisfying Hamilton's (canonical) Equations: $\frac{dq}{dt} = \frac{\partial H}{\partial p}$, $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$, $\frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial H}{\partial t}$; which can be rewritten as $\dot{z} = J\nabla H(t, z)$.
Hamiltonian System Advantage	Gives important insight about the dynamics, even if the initial value problem cannot be solved analytically. Example: 3BP, even if there is no simple solution to the general problem, Poincaré showed for first time that it exhibits deterministic chaos.
Constant of Motion VS Integrals of Motion/ First Integrals	In a force field COM is any function of time and phase-space coordinates that is constant throughout a trajectory (e.g., $C(x, v, t) = x - vt$) VS Functions of only the phase-space coordinates that are constant along an orbit.
Symplectic Matrix	$M \in \mathbb{R}^{2n \times 2n}$ that satisfies: $M^T \Omega M = \Omega$, where Ω is fixed $2n \times 2n$ nonsingular, skew-symmetric matrix. $\det M = 1$, & symplectic matrices in $\mathbb{R}^{2n\times 2n}$ form subgroup $Sp(2n, \mathbb{R})$ of special linear group $SL(2n, \mathbb{R})$ (set of matrices in $\mathbb{R}^{2n \times 2n}$ w/det 1)

Kepler Problem	2BP w/central force F that varies in strength as $\overrightarrow{F} = \frac{k}{r^2}\hat{r}$ Force may be attractive or repulsive. Solution can be expressed as a Kepler orbit using six orbital elements.
Kepler's Inverse Problem	Types of forces that would result in orbits obeying **Kepler's laws of planetary motion**
Kepler's Laws	Orbit of moving body (MB) is an ellipse with larger body (LB) at one of the two foci. Line segment joining MB & LB sweeps out equal areas during equal intervals of time. $(\text{orbital period of MB})^2 = k(\text{MBs semi-major axis})^3$, for some $k \in \mathbb{R}^+$.
Orbital Elements Kepler's	Shape and Size: Eccentricity (e), Semimajor axis (a) Orientation of Orbital Plane: Inclination (i), Longitude of ascending node (Ω) Remaining : Argument of periapsis (ω), True anomaly (ν, θ, or f) at epoch (t_0)
Orbital Elements: **Shape and Size**	**Eccentricity** (e): shape of ellipse, how elongated compared to circle. $\{0,(0,1),1,(1,\infty)\}$ **Semimajor axis** (a): $\frac{\text{periapsis} + \text{apoapsis}}{2}$. Means distance 'tween a focus & max dist. of orbit. For 2BP, is distance tween centers of the bodies, not distance of bodies from COM.
Orbital Elements: **Orientation of Orbital Plane**	**Inclination** (i): vertical tilt of ellipse measured @ascending nde. Tilt angle measured ⊥ to line of intersectn tween orbital & ref. plane. **Longitude of ascndng node** (Ω): horizontally orients ascndng node of ellipse wrt reference frame's vernal pt
Orbital Elements: **Remaining**	**Argument of periapsis** (ω): orientation of ellipse in orbital plane, as angle measured from ascending node to periapsis.**True anomaly** (θ) **at epoch** (t_0): position of body along ellipse at a specific time (the "epoch")
Kepler Problem Mathematically	Central force $\overrightarrow{F}(q)$ varies as: $\overrightarrow{F} = \frac{k}{r^2}\hat{r}$, where $r = \lvert q \rvert$, $\hat{r} = \frac{q}{\lvert q \rvert}$. Scalar potential energy of the non-central body is: $V(r) = -\frac{k}{r}$ Solve $\dot{q} = p$ & $\dot{p} = -k\frac{q}{\lvert q \rvert^3}$. ∃Sols on $\mathbb{R}^2 \backslash \Delta$. Can regularize for 2BP
3BP Existence & Uniqueness	Problems arise when there are collisions causing singularities in the differential equations. One can regularize double collisions, but not triple collisions.

Polar Coordinate Acceleratn	$\vec{a} = \vec{v}' = \left(\dot{r}\ (\cos\varphi,\ \sin\varphi) + r\ \dot{\varphi}\ (-\sin\varphi,\ \cos\varphi)\right)'$ $= \ddot{r}\ (\cos\varphi,\ \sin\varphi) + 2\ \dot{r}\dot{\varphi}\ (-\sin\varphi,\ \cos\varphi) + r\ \ddot{\varphi}\ (-\sin\varphi,\ \cos\varphi) - r\ \dot{\varphi}^2\ (\cos\varphi,\ \sin\varphi)$ Let $\hat{r} := (\cos\varphi,\ \sin\varphi)$, and $\hat{\varphi} := \frac{d\hat{r}}{d\varphi}$, then $\vec{v} = \dot{r}\ \hat{r} + r\ \dot{\varphi}\ \hat{\varphi}$, & $\vec{a} = \left(\ddot{r} - r\ \dot{\varphi}^2\right)\hat{r} + \left(2\ \dot{r}\dot{\varphi} + r\ \ddot{\varphi}\right)\hat{\varphi}$
Newton's Laws of Motion & Law of Universal Gravitation	1. $\Sigma\vec{F} = 0 \Leftrightarrow \frac{d\vec{v}}{dt} = 0$ (No External Force ⇔ No acceleration) 2. $\vec{F} = m\vec{a}$. 3. $\vec{F}_{12} = -\vec{F}_{21}$ $\vec{F} = -G\frac{Mm}{r^2}\hat{r}$
Solving 2BP Newton	Plug Polar into $\ddot{x} = -\frac{GM}{r^2}\hat{r}$, where $\left\lvert\vec{x}\right\rvert =: r$ and $\hat{r} = \frac{x}{\lvert x\rvert}$ separate components, using L, solve for $\dot{\varphi}$, plug back in. Result: $\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2}$. Then, c.o.c. $r \to \frac{L^2}{GMu} \to$ Linear Nonhomogeneous DEQ
Solving 2BP Lagrangian	Form Lagrangian $\mathcal{L} = T - U$. Euler-Lagrange **EOM**: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0$. Using L, solve for $\dot{\varphi}$, plug back in. Result: $\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2}$. Then, c.o.c. $r \to \frac{L^2}{GMu} \to$ Linear Nonhomogeneous DEQ
Define Conservative Force wrt Potential	Negative vector gradient of a potential field: $\vec{F}\left(\vec{r}\right) = -\nabla U = -\frac{dU}{d\vec{r}}$.
Gravitational Potential of particle *m* attracted to *M*	$U\left(\vec{r}\right) = -\int_{\infty}^{\vec{r}}\vec{F}\cdot d\vec{r}$ $= -\int_{\infty}^{\vec{r}} -\frac{GMm}{r^2}\hat{r}\cdot d\vec{r} = -\frac{GMm}{r}$.
Gravitational Kinetic of particle *m*	$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, or in polar coordinates: $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\ \dot{\theta}^2\right)$.
Newtonian NBP EOM	$m_i\frac{d^2q_i}{dt^2} = -\sum_{j=1, j\neq i}^{n}\frac{Gm_im_j(q_j-q_i)}{\lvert q_j-q_i\rvert^3} = -\frac{\partial U}{\partial q_i}$ $U := -\sum_{1\leq i<j\leq n}\frac{Gm_im_j}{\lvert q_j-q_i\rvert}$. System of $3n$ second order ODEs, with $6n$ initial conditions as 3n position and 3n momentum

Hamiltonian **NBP EOM** **H = T + U**	$p_i := m_i \frac{dq_i}{dt}$. Kinetic energy is $T = \sum_{i=1}^{n} \frac{1}{2} m_i v^2 = \sum_{i=1}^{n} \frac{\lvert p_i \rvert^2}{2m_i}$ $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$. Hamilton's equations show that the n-body problem is a system of 6n first-order differential equations.
Symmetries **in NBP**	**Translational symmetry**: $C = \frac{\Sigma m_i q_i}{\Sigma m_i}$, so $C = L_0 t + C_0$. (6 constants) **Rotational symmetry**: $A = \Sigma(q_i \times p_i)$. (3 constants) **Conservation of energy** H. Hence, every n-body problem has ten integrals of motion.
Scaling **invariance** **in NBP**	Because T and U are homogeneous functions of degree 2 and -1, respectively the equations of motion have a **scaling invariance**: if $q_i(t)$ is a solution, then so is $\lambda^{-\frac{2}{3}} q_i(\lambda t)$ for any $\lambda > 0$.
Prove COM **constant** **in NBP**	2nd Law: $m_i \ddot{r}_i = F_i$, 3rd Law: $\sum_i F_i = 0$. Summing over i: $\frac{d^2}{dt^2}(\sum_i m_i r_i) = \sum_i F_i = 0$. So $\frac{\sum_i m_i r_i}{\sum_i m_i} = c_1 t + c_2$, but by symmetry of translation invariance we can choose a moving inertial reference frame such that $\frac{\sum_i m_i r_i}{\sum_i m_i} = 0$.
Prove Energy is **constant** **in NBP**	$F_i := -\frac{d}{dr_i} U(r_1, r_2, \ldots)$. Then take total energy: $E = \sum_i \frac{m_i \lvert \dot{r}_i \rvert^2}{2} + U$. And differentiate with respect to time: $\frac{dE}{dt} = \sum_i m_i (\ddot{r}_i \dot{r}_i) + \sum_i \frac{dU}{dr_i} \dot{r}_i = \Sigma_i (F_i - F_i) \dot{r}_i = 0$.
Central Force **on a particle** **of mass** m	Force is always directed from m toward, or away, from a fixed point O Magnitude of the force only depends on the distance r from O $\vec{F}$ is C.F. $\Leftrightarrow \vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}$.
Particle moving **thru/Central Force** **Properties**	Path of particle must be a plane curve. Angular momentum of particle is conserved. Position vector sweeps out equal areas in equal times. (**Law of Areas**)
Conservative **Force** $\vec{F}$	Work $W = \int_A^B \vec{F} \cdot d\vec{r}$ done in moving from A→B is independent of path chosen. Only depends on the endpoints. So W from A assigns scalar value to every other point. Defines scalar potential field V. Force **defined** as $\vec{F}(\vec{r}) = -\frac{dV}{d\vec{r}}$. So $W = V(A) - V(B)$

How to compute potential V in central force field f?	$\vec{F}(\vec{r}) := -\frac{dV}{dr} \Rightarrow \vec{F} \cdot d\vec{r} = -dV$ (*). $\vec{r} \cdot \vec{r} = r^2$, and $d(\vec{r} \cdot \vec{r}) = d(r^2) \Rightarrow (\vec{r} \cdot d\vec{r}) + (d\vec{r} \cdot \vec{r}) = 2rdr$. So we have: $\vec{r} \cdot d\vec{r} = rdr$. From (LHS of *): $\vec{F} \cdot d\vec{r} = f(r)\frac{\vec{r}}{r} \cdot d\vec{r} = f(r)dr$. So, $f(r)dr = -dV \Rightarrow V = -\int f(r)dr$.

2BP Constant Areal Velocity "Law of Areas"	$\dot{A} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}\left\vert \vec{r} \times \frac{\Delta r}{\Delta t} \right\vert = \frac{1}{2}\left\vert \vec{r} \times \vec{v} \right\vert$. $\vec{r} \times \vec{v} = \vec{r} \times \left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right) = r\,\dot{r}\,(\hat{r} \times \hat{r}) + r^2\,\dot{\theta}\,\left(\hat{r} \times \hat{\theta}\right) = r^2\,\dot{\theta}\,\vec{k}$. $2\,\dot{A} = \left\vert \vec{r} \times \vec{v} \right\vert = r^2\,\dot{\theta} = \frac{\vert \vec{L} \vert}{m}$. $\dot{\vec{A}} = \dot{A}\,\vec{k} = \frac{1}{2} r^2\,\dot{\theta}\,\vec{k} = \frac{L}{2m}\vec{k}$ is constant **areal velocity**

Orbit Space	System after quotienting out of the orbit angle.

Formally stable relative equilibrium	evolutions of sufficiently small perturbations of RE solutions are arbitrarily confined to that relative equilibrium's orbit

Central Configuration Relative Equilibrium Relationship	Given the correct initial velocities, a central configuration will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.

Barycenter	$\vec{R} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$, position of the center of mass

Prove $\frac{d}{dt}$COM constant in 2BP	$\vec{R} := \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$, COM. Add force equations: $\vec{F}_{12} + \vec{F}_{21} = m_1\,\ddot{x}_1 + m_2\,\ddot{x}_2 = (m_1 + m_2)\,\ddot{R} = 0$ (by Newton's 3rd). $\ddot{R} = 0 \Rightarrow V = d\vec{R}/dt$ of COM is constant. So, **total momentum** $m_1 v_1 + m_2 v_2$ is also constant (**conserv. of momentum**).

2BP Solution Factoids	Two bodies' orbits are similar conic sections (differ by a ratio). The same ratios apply for the velocities, and, without the minus, for the angular momentum and for the kinetic energies, all with respect to the barycenter.

True Anomaly	Angle between the current position of the orbiting object and the location in the orbit at which it is closest to the central body (called the periapsis)

Newton's law of motion for the gravitational N-body problem	$m_i \ddot{q}_i = F_i = -\sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{r_{ij}^3}$ $F_i = -\nabla_i U$ where: $U = -\sum_{j \neq i} \frac{m_i m_j}{r_{ij}}$

Central Configuration Equation invariant under:	The Euclidean similarities of $\mathbb{R}^d$ translations, rotations, reflections and dilations.

Equivalent Central Configurations $q, q' \in \mathbb{R}^{Nd}$	There are constants $k \in \mathbb{R}$, $b \in \mathbb{R}^d$ and a $d \times d$ orthogonal matrix Q such that $q_i' = kQq_i + b,\ i = 1, \ldots, N$. So one can speak of an equivalence class of central configurations.

For configurations w/$c = 0$ (**COM at origin**), **the CC equations are**	$-\lambda q_i = \sum_{j \neq i} \frac{m_j (q_j - q_i)}{r_{ij}^3}$ and any configuration satisfying this equation has $c = 0$.

CC and 2BP	Any two configurations of $N = 2$ particles in $\mathbb{R}^d$ are equivalent. Every configuration of two bodies is central with $\epsilon \in [0, 1]$. Each mass moves on a conic section according to Kepler's laws.

Euler Collinear 3BP Sols	One equivalence class of collinear central configurations for each possible ordering of the masses along the line. Leading to periodic motions of all three bodies on ellipses.

Lagrange CC 3BP Sol	Equilateral triangle is CC for any 3 m_1, m_2, m_3. The only noncollinear CC for 3BP. Stable if $m_1 \geq 25 m_2$. Regular simplex is CC of N bodies in $N - 1$ dims for all choices of masses

Homothetic Motion	Released from rest, a CC maintains the same shape as it heads toward total collision

CC & Colliding Bodies	For any collision orbit in the nBP, the colliding bodies asymptotically approach a CC
Lagrangian point Def	5 Points near 2 large bodies in orbit where a smaller object will maintain position relative to large bodies. Forces of large bodies: centripetal & (for certain points) Coriolis match up
Stability of Lagrangian points	L4/L5 linearly stable if $\frac{M1}{M2}$ sufficiently large, where M_1 is the larger body. Kidney bean-shaped orbit around the point as seen in the corotating frame of reference. Nonlinearly stable via KAM
Coriolis Acceleration	"Ficticious Force." Depends on the velocity of an orbiting object and cannot be modeled as a contour map.Faster Angular Momentum⇒More Coriolis. Caused by Velocity perpendicular to rotational axis.
Orbits arising from Inverse Square Law	elliptical, parabolic or hyperbolic orbits.
NBP First Integrals	3 center of mass, 3 linear momentum, 3 angular momentum one for energy. Allows the reduction of system from **6n** variables to $6n - 10$.
Reduction of NBP Beyond first 10.	Beyond the 10 first integrals, **Jacobi** showed that using a so-called reduction of nodes (some symmetries), the dimension of the system could be further reduced to $6n - 12$.
Relative Equilibrium when $n = 1$	Steady rotations around the principal axes of inertia (found from the Moment of Inertia Matrix eigenvectors). Minimum energy motions are rotations around the **axis of maximum moment-of-inertia**.
When does Energetic Stability Occur? (**calculation**)	When the Hessian of the amended potential is positive definite or $[\partial_q^2 U_{red}]$, has only positive eigenvalues. With one negative eigenvalue, the system can escape from RE while conserving energy

When does the Hamiltonian represent the energy constant of motion?	If the Lagrangian (and therefore Hamiltonian) is not an explicity function of time. Often this is not the case in rotating reference frames.
Turning 2BP eq: $m\left(r\ddot{\varphi}+2\dot{r}\dot{\varphi}\right)=0$ **into a constant of motion**	$m\left(r\ddot{\varphi}+2\dot{r}\dot{\varphi}\right)=\frac{m}{r}\left(r^2\ddot{\varphi}+2r\dot{r}\dot{\varphi}\right)=\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\varphi}\right)=0$ or $r^2\dot{\varphi}=$ constant $=h$.
Way to show we have Conserved Total Energy w/NBP?	CC $\Rightarrow$ Force is a function of position only, so work over closed loops are zero, equivalently, work done between two points is independent of choice of path. $\Rightarrow$ Conserved Total Energy $\Rightarrow$ Conservative Force
Recall in 2BP: $L=mr^2\dot{\varphi}$ **So**, $r^2\dot{\varphi}=h$, **const**. **If** $h\neq 0$, **then**:	Curvilinear Sector of area swept out by $\vec{r}$ is: $S(t)=\frac{ht}{2}$, thus $\dot{S}=\frac{h}{2}$ and the sector velocity is constant. “area integral" or “Kepler’s 2nd Law". h is “area constant."
When is a force called conservative?	If there’s a potential V such that the components of force can be written as $F_i=-\frac{\partial V}{\partial x_i}\equiv-\partial_i V$. Gravity and electrostatic force satisfy this.
Pros of Lagrange Eqs vs Newton’s Laws of Motion	Lagrange’s EQ hold in arbitrary curvilinear coordinate system. # of Lagrange EQs = # of degrees of freedom. Newton: 3 EQs for each body & possibly constraint EQs
Derive Newton’s Force Law from Lagrange’s Equations	Euler-Lagrange EQ: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}-\frac{\partial\mathcal{L}}{\partial x_i}=0$. Observe: $\frac{\partial\mathcal{L}}{\partial\dot{x}_i}=m\dot{x}_i=p_i$. So, $\frac{dp_i}{dt}=\frac{\partial\mathcal{L}}{\partial x_i}$. Observe: $\frac{\partial\mathcal{L}}{\partial x_i}=\frac{\partial}{\partial x_i}(T-V)=-\frac{\partial V}{\partial x_i}$, since T does not depend on x_i. Observe: $F_i:=-\frac{\partial V}{\partial x_i}$, therefore $\frac{dp_i}{dt}=F_i$, Newton’s Law of Force.
Generalized Momentum conjugate to q_i **for Hamiltonian**	Defined to be: $p_i:=\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$ e.g., Angular Momentum $\vec{L}$
Legendre Transformation of $f(x_1,\ldots,x_n)\equiv f(x)$.	Let: $y_i=\frac{\partial f}{\partial x_i}$ and $g:=\Sigma x_i y_i-f$. g is the Legendre Transformation. Hamiltonian $\mathcal{H}=\Sigma p_i\dot{q}_i-\mathcal{L}$ is a transformation of $\mathcal{L}$.

Differences between Hamilton's Eqs & Lagrange's Eqs	Both hold in arbitrary curvilinear coordinate systems. Both EOM derived from scalar functions $\mathcal{L}$ or $\mathcal{H}$. $\mathcal{H}$: 1st-order, $\mathcal{L}$: 2nd-order. $\mathcal{L}$: One EQ per degree freedom. $\mathcal{H}$: Two EQ per degree freedom; one for q_i, and one for p_i.

How to Integrate Hamiltonian Problem	For an integral F, sols lie on $F^{-1}(c)$ with dim $= 2n - 1$. If you have $2n$–1 such independent ($\{F_i, F_j\} = 0,\ \forall i \neq j$) integrals F_i, then holding these fixed would define a !sol curve in $\mathbb{R}^{2n}$

Stable RE	An equilibrium point z_0 is stable if for every $\varepsilon > 0,\ \exists \delta > 0$ such that $\lvert z_0 - \varphi(t, z_1)\rvert < \varepsilon,\ \forall t$ whenever $\lvert z_0 - z_1\rvert < \delta$.

Define: $V : O \to \mathbb{R}$ as **pos**. **def**. **wrt f**.**p**. z_0 **of** $\dot{z} = f(z)$ **smooth**: If...	there is a neighborhood $Q \subset O$ of z_0 such that $V(z_0) < V(z),\ \forall z \in O \backslash \{z_0\}$. And, z_0 is called a strict local minimum of V.

Lyapunov's Stability Theorem	If there exists a function V that is positive definite wrt z_0 and such that $\dot{V} \leq 0$ in a neighborhood of z_0, then the equilibrium z_0 is positively stable (as $t \to \infty$).

Dirichlet's Stability Theorem	If z_0 is a strict local minimum or maximum of H, then z_0 is a stable equilibrium of $\dot{z} = J\nabla H(z)$.

Chetaev's Thm for $\dot{z} = J\nabla H(z) = f(z)$	$V : O \to \mathbb{R}$ a smooth function & Ω an open subset of O w/: $z_0 \in \partial\Omega$. Also: $V > 0$ for $z \in \Omega$. $V = 0$ for $z \in \partial\Omega$. $\dot{V} = V \cdot f > 0$ for $z \in \Omega$. Then, f.p. z_0 is unstable. $\exists N(z_0)$ such that sols in $N \cap \Omega$ leave N in positive time

Requirement for Generalized Coordinates	Span the space of the motion in phase space, and be linearly independent. Often found by: $p_i := \partial_{\dot{q}_i} \mathcal{L}$.

Requirements for Solving 2BP	2 Integrals of Motion (L, H) and two initial values (φ_0, r_0)

Poission Bracket of **F and G**	$\{F,G\} = \Sigma_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$. Skew-symmetric and bilinear In terms of phase variable $\vec{z}$, bracket of $F(\vec{z},t), G(\vec{z},t)$ is $\nabla F \cdot J\nabla G$ Associated with every Hamiltonian is a vector field defined by: $\hat{v}_H(F) = \{F,H\}$.

NBP. **How force on each mass $\vec{f}_i$ is derived given**: $\mathbf{V}\left(\vec{r}\right) = -\boldsymbol{\Sigma}_{i<j}\frac{Gm_im_j}{\lvert\vec{r}_j-\vec{r}_i\rvert}$.	$\vec{f}_i = -\nabla_{\vec{r}_i}V\left(\vec{r}\right) = \Sigma_{j\neq i}\vec{f}_{ij}\left(\vec{r}\right)$ where $\vec{f}_{ij} = \frac{Gm_im_j}{\lvert\vec{r}_j-\vec{r}_i\rvert^3}\left(\vec{r}_j - \vec{r}_i\right)$.

Potential for ERB. $V_{ij} =$	$dm_i = \rho_i\left(\vec{a}_i\right)d\vec{a}_i$, where $\vec{a}_i$ position in body frame $\mathcal{F}_i$ for $\mathcal{B}_i$ and ρ_i density distribution. $V_{ij} = -G\int_{\mathcal{B}_i} dm_i \int_{\mathcal{B}_j} dm_j \frac{1}{\lvert\left(\vec{r}_i-\vec{r}_j\right)+\mathbf{B}_i\vec{a}_i-\mathbf{B}_j\vec{a}_j\rvert}$, where $\mathbf{B}_i\left(\vec{\theta}_i\right)$ is the transformation matrix in Euler angles $\vec{\theta}_i = (\psi_i,\theta_i,\varphi_i)$ from body frame to inertial frame. So, $V\left(\vec{r},\vec{\theta}\right) = \Sigma_{i<j}V_{ij}$

"**Natural**" **Hamiltonian**	$H(q,p) = T(q,p) + U(q)$, where T is the kinetic energy, and U the potential energy. No time dependence.

Variational Equation for Equilibrium z_0	Assume $\delta z = z - z_0$ is infinitesimal, Variational Eq is $\delta\dot{z} = L\delta z$, where constant matrix $L = JD^2H(z_0)$ is the linearization. Solution is called the "tangent flow." Assuming distinct evals, it has the form: $\delta z = \Sigma_j c_j v_j e^{\sigma_j t}$, w/$\sigma_j$ evals & v_j evects

"**Hamiltonian Matrix**" L	$2n \times 2n$ matrix L such that JL is symmetric, where J is the Poisson matrix, and $L^TJ + JL = 0$. Example: $JD^2H(z_0) =: L$.

Eigenvalues of Hamiltonian Matrix	Come in pairs $\pm\sigma$. Therefore, Exponential growing terms exists unless all $\sigma \in i\mathbb{R}$. Thus, Linear Stability reduces to finding eigenvalues and eigenvectors of Hamiltonian Matrix L

Lyapunov Stability of Hamiltonian Systems	Equilibrium $z_0 \in \mathbb{R}^{2n}$ is Lyapunov stable (nonlinearly stable) if for every neighborhood V of z_0, there exists a neighborhood $U \subseteq V$ such that $z(0) \in U \Rightarrow z(t) \in V$ for all time.

Linear Stability of Hamiltonian Systems	F.p. $z_0 \in \mathbb{R}^{2n}$ is linearly stable if all orbits $z(t)$ of tangent flow are bounded $\forall t$. Thus, nonlinear much stronger than linear stability, as sets U & V where $z(t)$ begin don't have to be infinitesimally small. Need $\sigma \in i\mathbb{R}$ (like spectral), AND 1D Jordan blocks.

Spectral Stability of Hamiltonian Systems	Equilibrium $z_0 \in \mathbb{R}^{2n}$ is spectrally stable if $\sigma \in i\mathbb{R}$. If in addition, 1D Jordan blocks⇒Linearly stable.

Counterexample Linear Stability ⇏ NonLinear Stability	Cherry Hamilt.: $H = \frac{\omega_1}{2}(p_1^2+q_1^2) - \frac{\omega_2}{2}(p_2^2+q_2^2) - \frac{\alpha}{2}[2q_1p_1p_2 - q_2(p_1^2 - q_1^2)]$, At $(0,0)$, linearly stable $\left(\sigma_{1,2} = \pm i\omega_1 \ \& \ \sigma_{3,4} = \pm i\omega_2\right)$, when $\omega_2 = 2\omega_1$ an explicit solution shows nonlinear terms lead to explosive growth.

Orbital Stability of Hamiltonian	Describes the divergence of two neighboring orbits, regarded as point sets

Structural Stability of Hamiltonian	Sensitivity (or insensitivity) of the qualitative features (f.p. & invariant sets) of a flow to changes in parameters.

Hamiltonian Loss of Spectral Stability	$H(z,\mu)$ smooth in $\mu \Rightarrow \sigma$ also smooth in μ. Stability loss due to: $\sigma_{1,2} = \pm i\omega_1$ & $\sigma_{3,4} = \pm i\omega_2$ merge @0, & split onto $\mathbb{R}$ (saddle-nde). Or $\sigma_{1,2}$, $\sigma_{3,4}$ collide @$z_0, \overline{z}_0 \neq 0$ & split off into complex plane forming complex quadruplet (Krein bifurcation)

Hamiltonian Reduced Characteristic Polynomial	Since σ in ±pairs, characteristic polynomial P_{2n} is even. Introducing $\tau := -\sigma^2$ gives: $Q_n(\tau) = (-1)^n P_{2n} = \tau^n - A_1\tau^{n-1} + \ldots + (-1)^n A_n$.⇒ Hamiltonian f.p.s are spectrally stable ⇔ all zeros of $Q_n(\tau)$ are real positive. Use Sturm.

Sturm's Thm for polynml Q(τ)	Sequence: $\{F_k(\tau)\}$ by $F_0(\tau) := Q(\tau),\ F_1(\tau) := Q'(\tau)$. At each stage divide, $\frac{F_{k-2}}{F_{k-1}}$ to get G_{k-1} + Remainder = $G_{k-1} - \frac{F_k}{F_{k-1}}$, so $F_k = G_{k-1}F_{k-1} - F_{k-2}$, where $\deg F_k < \deg F_{k-1}$. $V(\tau)$:=(# of variations in sign). # of !(roots) in $(a,b]$ is $V(a) - V(b)$

Spectral Stability via Sturm's Thm	Recall for Hamiltonian stable zeros of Reduced $Q(\tau)$ must be nonnegative real. Via Sturm's Thm, this is true ⇔ $V(0) - V(\infty) = n$. For Natural systems, this implies nonlinear stability as well.

Lagrange-Dirichlet Theorem	Let the 2nd variation of the Hamiltonian $\delta^2 H$ be definite at an equilibrium z_0. Then, z_0 is stable. $\delta^2 H := \frac{d^2}{dt^2}H(z_0 + th)\vert_{t=0}$

Relative Equilibrium	Solution which becomes an equilibrium in some uniformly rotating a coordinate system. f.p. of dyn sys which has been reduced through quotienting out of rotation angle. Critical points of an "amended potential"
History RE for ERBs in F2BP	Maciejewski: 36 non-Lagrangian RE as $\vec{r} \to \infty$. Scheeres: Nec/Suff for pt/ERB. Moeckel: lower bounds on # of RE for F2BP where radius of the system is large, but finite
How to Reduce in orbital RE?	For RE, invariance of orbit requires uniform rot. w/fixed $\vec{L}$ & r. So, symmetry of φ about $\vec{L}$, not found in $\mathcal{L}$. Symmetry gives first integral & allows elimination of velocity variable by solving for it explicitly in EOM.
How solve general point mass 2BP	Change of variables such that 2BP→R2BP. $\vec{r} := \vec{r}_2 - \vec{r}_1$. $M := \frac{M_1 M_2}{M_1 + M_2}$. Then, apply sol. for Kepler Problem.
Central Force on m_i	The force on m_i is always directed toward, or away from a fixed point O; and The magnitude of the force only depends on the distance r of m_i from O.
Central Force Motion is Planar	init pos & vel vectors define a plane. $\vec{r} \cdot \vec{L} = \vec{r} \cdot \left(\vec{r} \times m\vec{v}\right) = m\vec{v} \cdot \left(\vec{r} \times \vec{r}\right) = 0$. $\vec{r}$ & $\frac{d\vec{r}}{dt}$ always lies in plane perpendicular to $\vec{L}$. $\vec{L}$ is constant$\Rightarrow$ $\vec{F}$ in plane. $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = (\vec{v} \times m\vec{v}) + \left(\vec{r} \times m\frac{d}{dt}\vec{v}\right) = \vec{r} \times \vec{F}$. And, CF$\Rightarrow$ $\vec{r} \times \vec{F} = 0$.
Derive Pot. from Newton law of gravitation tween m & M	$\vec{F}\left(\vec{r}\right) = -\frac{GMm}{r^3}\vec{r}$. Integrating we find: $U(r) = -\int_{\infty}^{\vec{r}} \vec{F}\left(\vec{s}\right) \cdot d\vec{s} = -\int_{\infty}^{\vec{r}} -\frac{GMm}{\lvert s \rvert^3}\vec{s} \cdot d\vec{s} = \int_{\infty}^{r} \frac{GMm}{s^2} ds$ $= -\frac{GMm}{r}$. And Kinetic is: $T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$
Find reduced Lagrangian $\mathbf{L}_{red}$ w/Red. EOM	$\mathcal{L}_{red} = T_{red} - U_{red}$, where T_{red} and U_{red} those necessary for $\frac{d}{dt}\frac{\partial \mathcal{L}_{red}}{\partial \dot{r}_i} - \frac{\partial \mathcal{L}_{red}}{\partial r_i}$ to produce reduced EOM
Usefulness of conservation of linear momentum	We can assume the system's COM moves at a constant rate. This allows us to choose an inertial reference frame such that our choice of origin coincides with the system's COM.

Dumbbell's Moment of Inertia	(Mass-of-point-1)(radius^2)+(Mass-of-point-1)(radius^2) = $(M_2 \boldsymbol{x}_1)\boldsymbol{x}_2^2 + (M_2 \boldsymbol{x}_2)\boldsymbol{x}_1^2 = M_2 \boldsymbol{x}_1 \boldsymbol{x}_2$ Scaled: $\frac{\boldsymbol{x}_1 \boldsymbol{x}_2}{M_1} = B$. Or $\boldsymbol{x}_{11}\boldsymbol{x}_{12} M_1 \ell_1^2 = \frac{\boldsymbol{x}_{11}\boldsymbol{x}_{12}\ell_1^2}{M_2} = B_1$
Simplify Equations with ratio variables $\boldsymbol{x}_1 + \boldsymbol{x}_2 = 1$	Let $\boldsymbol{x}_1 = \frac{u}{1+u}$ and $x_2 = \frac{1}{1+u}$. Note that we still have $\boldsymbol{x}_1 + \boldsymbol{x}_2 = 1$, but now we have characterized them with one variable $0 < u < \infty$.
Descartes' Rule of Signs for f	# of positive roots is at most the # of sign changes in sequence of f's coefficients (omitting zero coefficients), and that difference between these two #s is always even. This implies that if the # of sign changes is 0 or 1, then there are exactly 0 or 1 positive roots, resp.
Graph of $f' = f'' = 0$	Points in the space at which extremums and inflection points collide and annihilate $\backsim \rightarrow \diagdown$
Conic Sections	$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$, where one of A,B,C are non-zero. All circles are similar. 2 ellipses are similar $\Leftrightarrow$ ratios of lengths of minor axes to lengths of major axes are equal.

Chetaev's Thm for $\dot{z} = J\nabla H(z) = f(z)$	$V : O \to \mathbb{R}$ a smooth function & Ω an open subset of O w/: $z_0 \in \partial\Omega$. Also: $V > 0$ for $z \in \Omega$. $V = 0$ for $z \in \partial\Omega$. $\dot{V} = V \cdot f > 0$ for $z \in \Omega$. Then, f.p. z_0 is unstable. $\exists N(z_0)$ such that sols in $N \cap \Omega$ leave N in positive time

Cyclic Hamiltonian Coordinate φ	Doesn't appear in H. Momentum ($p = m\dot{\varphi}$) conjugate to φ is integral of motion. Associated w/symmetry of system. Noether identified correspondence. Generalized momentum $p = \frac{\partial L}{\partial \dot{q}}$, from Euler Lagrange $\frac{d}{dt}\partial_{\dot{q}}L = \partial_q L = 0$. So, p conserved

Relationship Between Work Force, and Potential	$W = \int_C \vec{F} \cdot d\vec{x} = \int_{\vec{x}(t_1)}^{\vec{x}(t_2)} \vec{F} \cdot d\vec{x} = U(\vec{x}(t_1)) - U(\vec{x}(t_2))$ If work for applied force is indep. of the path, then work done by (conservative) force, by the gradient theorem, defines a pot. funct.

Euler Angles **Axes**: xyz, XYZ **L.O.N.**: $N = z \times Z$	α is angle between x axis and N axis (Line of Nodes) β is angle between z axis and Z axis γ is angle between N axis and X axis

Levi-Civita Time Transfrmtn Step	$dt = rd\tau$ Adds variable to sys & DEQ, giving extended phase space. $\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{x'}{r}$. Substite in.

Levi-Civita Conformal Squaring Step	Represent the complex physical coordinate x as u^2 of $u = u_1 + iu_2$. So, $x = u^2$ parametric u-manifold is a Riemann surface w/2 sheets, connected by branch pts at $u = 0$ & $u = \infty$. $r = \lvert x\rvert = \lvert u\rvert^2 = u\overline{u}$. $x' = 2uu'$, $x'' = 2\left(uu'' + (u')^2\right)$, and $r' = u'\overline{u} + u\overline{u}'$

Levi-Civita Elimination Singularities Step	After Conformal Squaring. Produces linear DEQs for the unperturbed problem. $u'' + \frac{1}{2}u(\sim) = \lvert u\rvert^2\overline{u}f$, & $H = \sim$. Substitute H into DEQ: $u'' + \frac{1}{2}uH = \lvert u\rvert^2\overline{u}f$ $t' = r$, $H' = \langle x', f\rangle$. DEQs for the dependent vars u, t, H as functions of fictitious time τ.

Why can 2BP collisions be regularized	$\ddot{x} = -\alpha\lvert x\rvert^{-\alpha-2}$. $\exists$restriction on α: $\alpha = 2(1 - \frac{1}{n})$ for $n \in \mathbb{Z}^+$, for a body to be regularizable. And for Kepler problem, $\alpha = 1$ or $n = 2$.

Different Regularization Approaches	Sundman: didn't guarantee smoothness of flow wrt init data. Levi-Civita: ditches DEQ singularity. Guarantees info bout flows close to collisions. Easton: isolating block. Collision close orbit gives extn for collision orbit? (block regularization)

Block Regularzatn	$\dot{x} = y$ & $\dot{y} = -\alpha\lvert x\rvert^{-\alpha-2}x$, w/$\alpha > 0$. Let $x \to r^{\gamma}e^{i\theta}$, $y \to r^{-\beta\gamma}(v + iw)e^{i\theta}$, w\\$\beta = \frac{\alpha}{2}$ & $\gamma = \frac{1}{\beta+1}$ So: $\dot{r} = (\beta+1)v$, $\dot{\theta} = \frac{w}{r}$, $\dot{w} = \frac{\beta-1}{r}wv$, $\dot{v} = \frac{w^2+\beta(v^2-2)}{r}$ $\mathbf{M}=\left\{(r,\theta,w,v) : r \geq 0 \text{ \& DEQs Hold}\right\}$, $\mathbf{N}=\left\{(r,\theta,w,v)\in\mathbf{M}(h) : \vec{r} = 0\right\}$. N Reglbl$\Leftrightarrow$ $\beta = 1 - n^{-1}$

Bertrand's Theorem	For conservative central-force (CF) potentls w/bounded orbits, only 2 types of CF potentials w/property that {bounded orbits} = {closed orbits}: 1) inverse-square CF such as gravitational or electrostatic potentl: $V(r) = -\frac{k}{r}$, & (2) radial harmonic oscillator potential: $V(r) = \frac{1}{2}kr^2$

Bertrand Orbit Shape	closed orbits are all ellipses. In inverse square case, force is directed toward one focus of ellipse. In harmonic oscillator, force directed toward geometric center of ellipse.

Conservation of Linear Momentum in NBP	$m_k\ddot{x}_k = \sum_{j=1,\, j\neq k}^{n}\frac{m_j m_k}{r_{jk}^3}(x_j - x_k)$. Summing RHS gives zero. So, $\ddot{\rho} = \frac{d^2}{dt^2}\sum_{k=1}^{n} m_k x_k = 0$, or $\rho = L_0 t + \rho_0$. Expresses **translational symmetry** COM moves uniformly in straight line

Conservation of Energy H in NBP	$H := T - U.\ T := \frac{1}{2}\sum_{k=1}^{n} m_k \lvert v_k \rvert^2$. $F_i = -\frac{d}{dx_i}U(x_1, x_2, x_3)$. So differentiating H with respect to time: $\frac{dH}{dt} = \sum_i m_i(\ddot{x}_i\ v_i) + \sum_i \frac{dU}{dx_i} v_i = (F - F)v_i = 0.$
Conservation of Angular Momentum in NBP	$m_k\ \ddot{x}_k = \sum_{j=1,\ j\neq k}^{n} \frac{m_j m_k}{r_{jk}^3}(x_j - x_k)$, forming $x_k \times\ \ddot{x}_k$, and summing: $\sum_{k=1}^{n} m_k\left(x_k \times \ddot{x}_k\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{m_j m_k}{r_{jk}^3} x_k \times x_j = 0$. Integrating LHS: $\sum_{k=1}^{n} m_k(x_k \times v_k) = c$. Expresses rotational symmetry
Constant Areal Velocity	Approximate area of arc sweep by Parallelogram: $\dot{A} = \frac{1}{2}\lvert \vec{r} \times \vec{v} \rvert$. $\vec{r} \times \vec{v} = \vec{r} \times \left(\dot{r}\ \hat{r} + r\ \dot{\theta}\ \hat{\theta}\right) = r\ \dot{r}\ (\hat{r} \times \hat{r}) + r^2\ \dot{\theta}\ \left(\hat{r} \times \hat{\theta}\right) = r^2\ \dot{\theta}\ \vec{k}.$ Result: $\dot{\vec{A}} = \dot{A}\ \vec{k} = \frac{1}{2} r^2\ \dot{\theta}\ \vec{k}$. Kepler's 2nd law
Bertrand Proof $m\ \ddot{r} - mr\ \dot{\theta}^2 = -V_r$	EOM. Eliminate $\dot{\theta}$ w\L, & time w\ $\frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\theta}$. C.O.V. $u \equiv \frac{1}{r} \Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{d}{du}V(\frac{1}{u}) =: J$, quasilinear. Pert from circ: $\eta \equiv u - u_0$ into a J tayl series. Let $\beta^2 :\equiv 1 - J'(u_0)$. $\Rightarrow B \in \mathbb{Q}$, cuz $\eta \approx k\cos(\beta\theta)$ Fourier $\eta = h_0 + h_1 \cos\beta\theta + \ldots$, substitute in. Equate low frequency. Get $\beta^2(1-\beta^2)(4-\beta^2) = 0$