Dynamical Systems Notes

These are notes takento prepare for an oral exam in Dynamical Systems. The content comes from various sources, including from notes from a class taught by Arnd Scheel.

Overview - Definitions:

Dynamical System: Mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. The concept unifies very different types of such "rules" in mathematics: the different choices made for how time is measured and the special properties of the ambient space may give an idea of the vastness of the class of objects described by this concept. Time can be measured by integers, by real or complex numbers or can be a more general algebraic object, losing the memory of its physical origin, and the ambient space may be simply a set, without the need of a smooth space-time structure defined on it.

There are two classes of definitions for a dynamical system: one is motivated by ordinary differential equations and is geometrical in flavor; and the other is motivated by ergodic theory and is measure theoretical in flavor. The measure theoretical definitions assume the existence of a measure-preserving transformation. This appears to exclude dissipative systems, as in a dissipative system a small region of phase space shrinks under time evolution. A simple construction (sometimes called the Krylov–Bogolyubov theorem) shows that it is always possible to construct a measure so as to make the evolution rule of the dynamical system a measure-preserving transformation. In the construction a given measure of the state space is summed for all future points of a trajectory, assuring the invariance.

The difficulty in constructing the natural measure for a dynamical system makes it difficult to develop ergodic theory starting from differential equations, so it becomes convenient to have a dynamical systems-motivated definition within ergodic theory that side-steps the choice of measure.

A **dynamical system** is a manifold M called the **phase** (or **state**) **space** endowed with a family of smooth evolution functions Φ^t that for any element of $t \in T$, the time, maps a point of the phase space back into the phase space. The notion of smoothness changes with applications and the type of manifold. There are several choices for the set T. When T is taken to be the reals, the dynamical system is called a **flow**; and if T is restricted to the non-negative reals, then the dynamical system is a **semi-flow**. When T is taken to be the integers, it is a **cascade** or a **map**; and the restriction to the non-negative integers is a **semi-cascade**.

The **evolution function** Φ_t is often the solution of a differential **equation of motion**: $\dot{x} = v(x)$. The equation gives the time derivative of a trajectory $x(t)$ on the phase space starting at some point x_0. The vector field $v(x)$ is a smooth function that at every point of the phase space M provides the velocity vector of the dynamical system at that point. (These vectors are not vectors in the phase space M, but in the tangent space T_xM of the point x.) Given a smooth Φ^t, an autonomous vector field can be derived from it.

There's no need for higher order derivatives in the equation, nor for time dependence in $v(x)$, these can be eliminated by considering systems of higher dimension. Other types of differential equations than $\dot{x} = v(x)$ can be used to define the evolution: $G(x, \dot{x}) = 0$ is an example arising from modeling of mechanical systems with complicated constraints. The differential equations determining the evolution Φ_t are often ODEs; in this case the phase space M is a finite dimensional manifold.

Geometrical Definition

A dynamical system is the tuple $\langle \mathcal{M}, f, \mathcal{T} \rangle$, with $\mathcal{M}$ a manifold (locally a Banach space or Euclidean space), $\mathcal{T}$ the domain for time (non-negative reals, the integers, ...) and f an evolution rule $t \to f^t$ (with $t \in \mathcal{T}$) such that f^t is a diffeomorphism of the manifold to itself. So, f is a mapping of the time-domain $\mathcal{T}$ into the space of diffeomorphisms of the manifold to itself. In other terms, $f(t)$ is a diffeomorphism, for every time t in the domain $\mathcal{T}$.

Measure theoretical definition

A dynamical system may be defined formally, as a **measure-preserving transformation** of a **sigma-algebra**, the quadruplet (X, Σ, μ, τ). Here, X is a set, and Σ is a sigma-algebra on X, so that the pair (X, Σ) is a **measurable space**. μ is a **finite measure** on the sigma-algebra, so that the triplet (X, Σ, μ) is a **probability space**. A map $\tau : X \to X$ is said to be

Σ-**measurable** if and only if, for every $\sigma \in \Sigma$, one has $\tau^{-1}\sigma \in \Sigma$. A map τ is said to be a **measure preserving map** if and only if, for every $\sigma \in \Sigma$, one has $\mu(\tau^{-1}\sigma) = \mu(\sigma)$. Combining the above, a map τ is said to be a **measure-preserving transformation** of X , if it is a map from X to itself, it is Σ-measurable, and is measure-preserving. The quadruple (X, Σ, μ, τ), for such a τ, is then defined to be a **dynamical system**.

The map τ embodies the time evolution of the dynamical system. Thus, for discrete dynamical systems the iterates $\tau^n = \tau \circ \tau \circ \cdots \circ \tau$ for integer n are studied. For continuous dynamical systems, the map τ is understood to be a finite time **evolution map** and the construction is more complicated.

Examples:

Logistic map	**Complex quadratic polynomial**	**Dyadic transformation**
Tent map	**Double pendulum**	**Arnold's cat map**
Horseshoe map	**Baker's map**; **a chaotic piecewise linear map**	**Billiards and outer billiards**
Hénon map	**Lorenz system**	**Circle map**
Rössler map	**Kaplan–Yorke map**	

List of chaotic maps: Swinging Atwood's machine, Quadratic map simulation system, Bouncing ball dynamics.

Definitions

A dynamical system often has an evolution rule of the form: $x' = f(x,t),\ x \in \mathbb{R}^n,$ with an init. cond. $x(t_0) = x_0$. (1)

Lyapunov Stability: If the solutions that start out near an equilibrium point x_e stay near x_e forever, then x_e is Lyapunov stable.

Asymptotically Stable: if x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable.

Exponential Stability: An equilibrium point $x_e = 0$ is an exponentially stable equilibrium point of (1) if there exist constants $m, \alpha > 0$ and $\varepsilon > 0$ such that for all $|x(t_0)| \leq \varepsilon$ and $t \geq t_0$, we have: $|x(t)| \leq me^{-\alpha(t-t_0)}|x(t_0)|$. The largest constant α which may be utilized is called the **rate of convergence**.

Exponential $\subseteq$ **Asymptotic** $\subseteq$ **Lyapunov**

Structural Stability: Fundamental property of some dynamical systems which implies that the qualitative behavior of the trajectories are unaffected by small perturbations (to be exact, C^1-small perturbations). Examples of such qualitative properties are numbers of fixed points and periodic orbits (but not their periods). Unlike Lyapunov stability, which considers perturbations of initial conditions for a fixed system, structural stability deals with perturbations of the system itself.

Phase-Space Distribution Function: A function of seven variables, $f\left(x, y, z, t;\ v_x, v_y, v_z\right) = \rho(p, q; t)$, which gives the number of particles per unit volume in single-particle phase space. It is the number of particles per unit volume having approximately the velocity $v = (v_x, v_y, v_z)$ near the position $r = (x, y, z)$ and time t.

Liouville's Theorem: The phase-space distribution function is constant along the trajectories of the system—that is that the density of system points in the vicinity of a given system point traveling through phase-space is constant with time. This time-independent density is in statistical mechanics known as the classical **a priori probability**.

The Liouville Equation: Describes the time evolution of the phase space distribution function $\rho(p, q; t)$: $\frac{d\rho}{dt} = \frac{\partial \rho}{\rho t} + \sum_{i=1}^{n}\left(\frac{\partial \rho}{\partial q_i}\ \dot{q}_i\ + \frac{\partial \rho}{\partial p_i}\ \dot{p}_i\ \right)$. Time derivatives are evaluated according to Hamilton's equations for the system. This equation demonstrates the conservation of density in phase space. **Liouville's Theorem** states "The distribution function is constant along any trajectory in phase space."

Poincaré Recurrence Theorem: Any dynamical system defined by an ordinary differential equation determines a flow map f_t mapping phase space on itself. The system is said to be volume-preserving if the volume of a set in phase space is invariant under the flow. For instance, all Hamiltonian systems are volume-preserving because of Liouville's theorem. The theorem is then: If a flow preserves volume and has only bounded orbits, then for each open set there exist orbits that

intersect the set infinitely often.

Ergodic Theory: A central concern of ergodic theory is the behavior of a dynamical system when it is allowed to run for a long time. The first result in this direction is the Poincaré recurrence theorem, which claims that almost all points in any subset of the phase space eventually revisit the set. More precise information is provided by various ergodic theorems which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. For **ergodic systems**, this time average is the same for almost all initial points: statistically speaking, the system that evolves for a long time "forgets" its initial state.

Bifurcation Theory: Mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. A **bifurcation** occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior.

Sharkovsky's Theorem (Re: periods of discrete dynamical systems): Implication: if a discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period.

Linear dynamical systems

Linear Dynamical Systems can be solved in terms of simple functions and the behavior of all orbits classified. In a linear system the phase space is the N-dimensional Euclidean space, so any point in phase space can be represented by a vector with N numbers. The analysis of linear systems is possible because they satisfy a superposition principle: if $u(t)$ and $w(t)$ satisfy the differential equation for the vector field (but not necessarily the initial condition), then so will $u(t) + w(t)$.

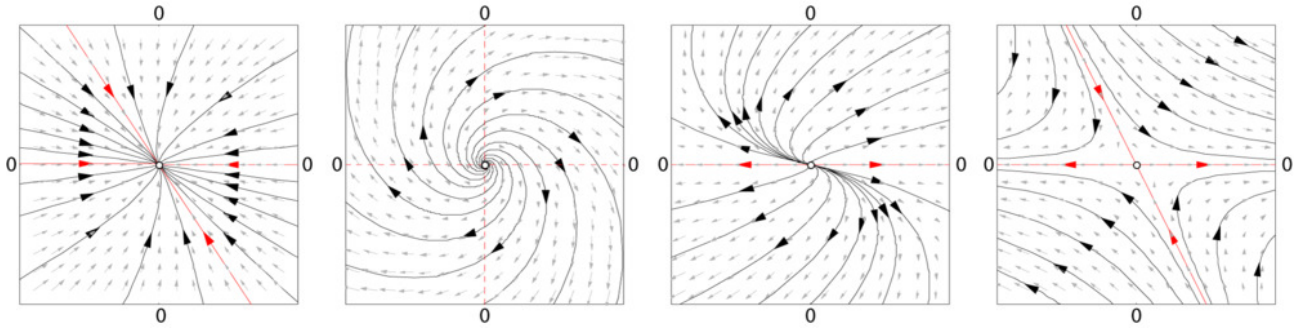
Flows

For a **flow**, the vector field $\Phi(x)$ is an affine function of position in phase space, that is: $\dot{x} = \phi(x) = Ax + b$. The solution to this system can be found by using the **superposition principle** (linearity). The case $b \neq 0$ with $A = 0$ is just a straight line in the direction of b: $\Phi^t(x_1) = x_1 + bt$.

When b is zero and $A \neq 0$ the origin is an equilibrium (or singular) point of the flow. For other initial conditions, the equation of motion is given by the exponential of a matrix: for an initial point x_0: $\Phi^t(x_0) = e^{tA}x_0$.

When $b = 0$, eigenvalues of A determine the structure of the phase space. From eigenvalues and eigenvectors of A it's possible to determine if an initial point will converge or diverge to the equilibrium point at the origin.

The distance between two different initial conditions in the case $A \neq 0$ will change exponentially in most cases, either converging exponentially fast towards a point, or diverging exponentially fast. Linear systems display sensitive dependence on initial conditions in the case of divergence. For nonlinear systems this is one of the (necessary but not sufficient) conditions for chaotic behavior.



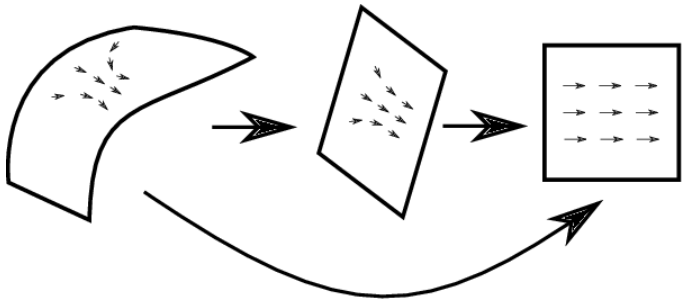
Maps

A discrete-time, **affine** dynamical system has the form of a **matrix difference equation**: $x_{n+1} = Ax_n + b$, with A a matrix and b a vector. As in the continuous case, the change of coordinates $x \to x + (1 - A)^{-1}b$ removes the term b from the equation. In the new coordinate system, the origin is a fixed point of the map and the solutions are of the linear system $A^n x_0$. The solutions for the map are no longer curves, but points that hop in the phase space. The orbits are organized in curves, or fibers, which are collections of points that map into themselves under the action of the map.

As in the continuous case, the eigenvalues and eigenvectors of A determine the structure of phase space. For example, if u_1 is an eigenvector of A, with a real eigenvalue smaller than one, then the straight lines given by the points along αu_1, with $\alpha \in \mathbb{R}$, is an invariant curve of the map. Points in this straight line run into the fixed point. There are also many other discrete dynamical systems.

Local Dynamics

The **qualitative properties** of dynamical systems do not change under a smooth change of coordinates (this is sometimes taken as a definition of qualitative): a singular point of the vector field (a point where $v(x) = 0$) will remain a singular point under smooth transformations; a **periodic orbit** is a loop in phase space and smooth deformations of the phase space cannot alter it being a loop. It is in the neighborhood of singular points and periodic orbits that the structure of a phase space of a dynamical system can be well understood. In the qualitative study of dynamical systems, the approach is to show that there is a change of coordinates (usually unspecified, but computable) that makes the dynamical system as simple as possible.



Rectification

A flow in most small patches of the phase space can be made very simple. If y is a point where the vector field $v(y) \neq 0$, then there is a change of coordinates for a region around y where the vector field becomes a series of parallel vectors of the same magnitude. This is known as the **Rectification Theorem**.

The rectification theorem says that away from singular points the dynamics of a point in a small patch is a straight line. The patch can sometimes be enlarged by stitching several patches together, and when this works out in the whole phase space M the dynamical system is **integrable**. In most cases the patch cannot be extended to the entire phase space. There may be singular points in the vector field (where $v(x) = 0$); or the patches may become smaller and smaller as some point is approached. The more subtle reason is a **global constraint**, where the trajectory starts out in a patch, and after visiting a series of other patches comes back to the original one. If the next time the orbit loops around phase space in a different way, then it is impossible to rectify the vector field in the whole series of patches, i.e. not integrable.

Poincaré map

A first recurrence map or Poincaré map, named after Henri Poincaré, is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower-dimensional subspace, called the Poincaré section, transversal to the flow of the system. More precisely, one considers a periodic orbit with initial conditions within a Poincaré section of the state space, which then leaves the section, and one observes the location at which the orbit first returns to the Poincaré section. One then creates a map sending the first point to the second, hence the name **first recurrence map**. The transversality of the Poincaré section means that periodic orbits starting on the subspace flow through it and not parallel to it.

A Poincaré map can be interpreted as a discrete dynamical system with a state space one dimension smaller than the original continuous one. Because it preserves many properties of periodic and quasiperiodic orbits of the original system and has a lower-dimensional state space, it is often used for analyzing the original system in a simpler way. In practice this is not always possible as there's no general method to construct a Poincaré map.

A Poincaré map differs from a **recurrence plot**, in that space (not time) determines when to plot a point. For instance, the locus of the Moon when the Earth is at perihelion is a recurrence plot; the locus of the Moon when it passes through the plane perpendicular to the Earth's orbit and passing through the Sun and the Earth at perihelion is a Poincaré map. It was used by Michel Hénon to study the motion of stars in a galaxy, because the path of a star projected onto a plane looks like a tangled mess, while the Poincaré map shows the structure more clearly.

Let $(\mathbb{R}, M, \varphi)$ be a global dynamical system, M the phase space and φ the evolution function. Let γ be a periodic orbit through a point x_0 and S be a local differentiable and transversal section of φ through x_0, called a Poincaré section through x_0. Given an open and connected neighborhood $U \subset S$ of x_0, a function: $P : U \to S$ is called Poincaré map for the orbit γ on the Poincaré section S through the point x_0 if:

- $P(x_0) = x_0$
- $P(U)$ is a neighborhood of x_0 and $P : U \to P(U)$ is a diffeomorphism
- for every point x in U, the positive semi-orbit of x intersects S for the first time at $P(x)$

Poincaré Maps and Stability Analysis

Poincaré maps can be interpreted as a discrete dynamical system. The stability of a periodic orbit of the original system is closely related to the stability of the fixed point of the corresponding Poincaré map.

Let $(\mathbb{R}, M, \varphi)$ be a differentiable dynamical system with periodic orbit γ through x_0. Let $P : U \to P(U)$, be the corresponding Poincaré map through x_0. We define: $P^0 := id_U$, $P^{n+1} := P \circ P^n$, $P^{-n-1} := P^{-1} \circ P^{-n}$, and $P(n, x) := P^n(x)$, then $(\mathbb{Z}, U, P)$ is a discrete dynamical system with state space U and evolution function $P : \mathbb{Z} \times U \to U$. By definition, this system has a fixed point at x_0. The periodic orbit γ of the continuous dynamical system is stable if and only if the fixed point x_0 of the discrete dynamical system is stable. The periodic orbit γ of the continuous dynamical system is asymptotically stable if and only if the fixed point x_0 of the discrete dynamical system is asymptotically stable.

Near periodic orbits

In general, in the neighborhood of a periodic orbit the rectification theorem cannot be used. Poincaré developed an approach that transforms the analysis near a periodic orbit to the analysis of a map. Pick a point x_0 in the orbit γ and consider the points in phase space in the neighborhood that are perpendicular to $v(x_0)$. These points are a **Poincaré section** $S(\gamma, x_0)$, of the orbit. The flow now defines a map, the **Poincaré map** $P : S \to S$, for points starting in S and returning to S. Not all these points will take the same amount of time to come back, but the times will be close to the time it takes x_0.

The intersection of the periodic orbit with the Poincaré section is a fixed point of the Poincaré map P. By a translation, the point can be assumed to be at $x = 0$. The Taylor series of the map is $P(x) = J \cdot x + O(x_2)$, so a change of coordinates h can only be expected to simplify P to its linear part: $h^{-1} \circ P \circ h(x) = J \cdot x$.

This is known as the **conjugation equation**. Finding conditions for which this equation holds has been one of the major tasks of research in dynamical systems. Poincaré first approached it assuming all functions to be analytic and in the process discovered the **non-resonant condition**. If $\gamma_1, \ldots, \gamma_\nu$ are the eigenvalues of J, they will be resonant if one eigenvalue is an integer linear combination of two or more of the others. Avoiding terms of the form γ_i-Σ (multiples of

other eigenvalues) occuring in the denominator of the terms for the function h, is called the non-resonant condition. It is also known as the **small divisor problem**.

Conjugation results

The results on the existence of a solution to the conjugation equation depend on the eigenvalues of J and the degree of smoothness required from h. As J does not need to have any special symmetries, its eigenvalues will typically be complex numbers. When the eigenvalues of J are not on the unit circle, the dynamics near the fixed point x_0 of P are called **hyperbolic** and when the eigenvalues are on the unit circle and complex, the dynamics are called **elliptic**.

In the hyperbolic case, the **Hartman–Grobman Theorem** gives the conditions for the existence of a continuous function that maps the neighborhood of the fixed point of the map to the linear map $J \cdot x$. The hyperbolic case is also structurally stable. Small changes in the vector field will only produce small changes in the Poincaré map and these small changes will reflect in small changes in the position of the eigenvalues of J in the complex plane, implying that the map is still hyperbolic.

Hartman–Grobman Theorem (**hyperbolic linearization theorem**): Deals with the local behavior of dynamical systems in the neighbourhood of a hyperbolic equilibrium point. It asserts that linearization — a natural simplification of the system — is effective in predicting qualitative patterns of behavior. The theorem states that the behavior of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point, where hyperbolicity means that no eigenvalue of the linearization has real part equal to zero. Therefore, when dealing with such dynamical systems one can use the simpler linearization of the system to analyze its behavior around equilibria.

Consider a system evolving in time with state $x(t) \in \mathbb{R}^n$ that satisfies the differential equation $\dot{x} = f(x)$ for some smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose the map has a hyperbolic equilibrium state $x_0 \in \mathbb{R}^n$: that is, $f(x_0) = 0$ and the Jacobian matrix $J = [\frac{\partial f_i}{\partial x_j}]$ of f at state x_0 has no eigenvalue with real part equal to zero. Then there exists a neighborhood N of the equilibrium x_0 and a homeomorphism $h : N \to \mathbb{R}^n$, such that $h(x_0) = 0$ and such that in the neighbourhood N the flow of $\dot{x} = f(x)$ is topologically conjugate by the continuous map $U = h(x)$ to the flow of its linearization $dU/dt = JU$.

Quasiperiodic Motion: Type of motion executed by a dynamical system containing a finite number (two or more) of incommensurable frequencies. That is, if we imagine the phase space is modelled by a torus T (that is, variables are periodic like angles), the trajectory of the system is modelled by a curve on T that wraps around the torus without ever exactly coming back on itself. A quasiperiodic function on the real line is the type of function (continuous, say) obtained from a function on T, by means of a curve: $\mathbb{R} \to T$, which is linear (when lifted from T to its covering Euclidean space), by composition. It's therefore oscillating, with a finite number of underlying frequencies.

Small Divisors: Expressions of form $\gamma \cdot k = \sum_{i=1}^{d} \gamma_i k_i$ with $k \in \mathbb{Z}^d \setminus \{0\}$, which usually are related to Fourier modes associated to the perturbing function and where the frequency vector γ often depends upon the slow (action) variables. Such expressions appear in the denominator of (formal) Fourier expansions of the object one aims to construct (e.g., formal expansion of a quasi-periodic motion). Since $\gamma \cdot k$ may became arbitrarily small for any vector $\gamma \in \mathbb{R}^d$ as k varies, the convergence of the perturbative series is in doubt (the **small-divisor problem**).

Kolmogorov–Arnold–Moser (**KAM**) **Theorem** (**elliptic theorem**): A useful paraphrase of KAM theorem is, "For sufficiently small perturbation, almost all tori (excluding those with rational frequency vectors) are preserved." KAM gives behavior near an elliptic point. KAM is a result about persistence of quasiperiodic motions under small perturbations. The theorem partly resolves the small-divisor problem that arises in perturbation theory of classical mechanics.

The problem is whether or not a small perturbation of a conservative dynamical system results in a lasting quasiperiodic orbit. Arnold originally thought that this theorem could apply to the motions of the solar system or other instances of the n-body problem, but Arnold's formulation of the problem turned out to work only for the three-body problem because of a degeneracy for larger numbers of bodies. Later, Gabriella Pinzari showed how to eliminate this degeneracy by developing a rotation-invariant version of the theorem.

The KAM theorem is usually stated in terms of trajectories in phase space of an integrable Hamiltonian system. The motion of an integrable system is confined to an invariant torus (a doughnut-shaped surface). Different initial conditions of the integrable Hamiltonian system will trace different invariant tori in phase space. Plotting the coordinates of an

integrable system would show that they are quasiperiodic.

The KAM theorem states that if the system is subjected to a weak nonlinear perturbation, some of the invariant tori are deformed and survive, while others are destroyed. Surviving tori meet the non-resonance condition, i.e., they have "sufficiently irrational" frequencies. This implies that the motion continues to be quasiperiodic, with the independent periods changed (as a consequence of the non-degeneracy condition). The KAM theorem quantifies the level of perturbation that can be applied for this to be true. Those KAM tori that are destroyed by perturbation become invariant Cantor sets, named Cantori by Ian C. Persival in 1979.

The non-resonance and a non-degeneracy conditions of the KAM theorem become increasingly difficult to satisfy for systems with more degrees of freedom. As the number of dimensions of the system increases, the volume occupied by the tori decreases. As the perturbation increases and the smooth curves disintegrate we move from KAM theory to **Aubry–Mather theory** which requires less stringent hypotheses and works with the Cantor-like sets.

The existence of a KAM theorem for perturbations of quantum many-body integrable systems is still an open question, although it is believed that arbitrarily small perturbations will destroy integrability in the infinite size limit. An important consequence of the KAM theorem is that for a large set of initial conditions the motion remains perpetually quasiperiodic.

Bifurcation theory

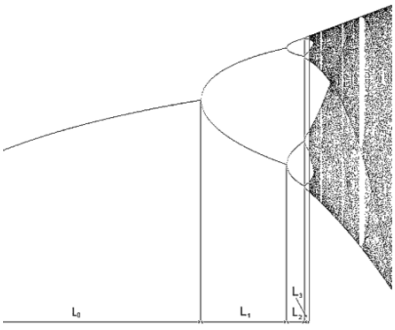
When the evolution map Φ_t (or the vector field it is derived from) depends on a parameter μ, the structure of the phase space will also depend on this parameter. Small changes may produce no qualitative changes in the phase space until a special value μ_0 is reached. At this point the phase space changes qualitatively and the dynamical system is said to have gone through a bifurcation.

Bifurcation Theory considers a structure in phase space (typically a fixed point, a periodic orbit, or an invariant torus) and studies its behavior as a function of the parameter μ. At the bifurcation point, the structure may change its stability, split into new structures, or merge with other structures. By using Taylor series approximations of the maps and an understanding of the differences that may be eliminated by a change of coordinates, it is possible to catalog the bifurcations of dynamical systems.

The bifurcations of a hyperbolic fixed point x_0 of a system family F_μ can be characterized by the eigenvalues of the first derivative of the system $DF_\mu(x_0)$ computed at the bifurcation point. For a map, the bifurcation will occur when there are eigenvalues of DF_μ on the unit circle. For a flow, it will occur when there are eigenvalues on the imaginary axis.

Some bifurcations can lead to very complicated structures in phase space. For example, the **Ruelle–Takens** scenario describes how a periodic orbit bifurcates into a torus and the torus into a strange attractor. In another example, **Feigenbaum period-doubling** describes how a stable periodic orbit goes through a series of period-doubling bifurcations.

Feigenbaum Period-Doubling



Feigenbaum constants are two mathematical constants which both express ratios in a bifurcation diagram for a non-linear **map**. They are named after the physicist Mitchell Feigenbaum. Feigenbaum originally related the first constant to the period-doubling bifurcations in the logistic **map**, but also showed it to hold for all one-dimensional maps with a single quadratic maximum. As a consequence of this generality, every chaotic system that corresponds to this

description will bifurcate at the same rate.

The **first Feigenbaum constant** is the limiting ratio of each bifurcation interval to the next between every period doubling, of a one-parameter map: $x_{i+1} = f(x_i)$, where $f(x)$ is a function parameterized by the bifurcation parameter r. It is given by the limit: $\delta = \lim_{n\to\infty} \frac{r_{n-1}-r_{n-2}}{r_n-r_{n-1}} = 4.669201609\ldots$, where r_n are discrete values of r at the n-th period doubling.

The **second Feigenbaum constant**, $\alpha = 2.502907875095892822283902873218\ldots$, is the ratio between the width of a tine and the width of one of its two subtines (except the tine closest to the fold). A negative sign is applied to α when the ratio between the lower subtine and the width of the tine is measured. These numbers apply to a large class of dynamical systems (for example, dripping faucets to population growth).

Logistic Map: A polynomial mapping (equivalently, recurrence relation) of degree 2, often cited as an archetypal example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations. The map was popularized as a discrete-time demographic model analogous to the logistic equation. Written $x_{n+1} = rx_n(1 - x_n)$, where x_n is a number between zero and one that represents the ratio of existing population to the maximum possible population. The values of interest for the parameter r are those in the interval [0,4]. This nonlinear difference equation is intended to capture two effects:

- reproduction where the population will increase at a rate proportional to the current population when the population size is small.
- starvation (density-dependent mortality) where the growth rate will decrease at a rate proportional to the value obtained by taking the theoretical "carrying capacity" of the environment less the current population.

However, as a demographic model the logistic map has the pathological problem that some initial conditions and parameter values (for example, if r > 4) lead to negative population sizes.

Floquet theory

So with Poincaré Maps, we have a theory which dealt above with one-dimensional periodic orbits, and now we turn to a theory which helps us with periodic orbits of higher dimension. So, say we are given a system $\dot{x} = f(x)$ with $\vec{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^n$. And assume that the system has a periodic orbit of period T where $\Gamma : x = \gamma(t), \quad 0 \le t \le T$, contained in E. The derivative of the Poincaré map $D\vec{P}(x_0)$, at a point $x_0 \in \Gamma$ is an $(n-1) \times (n-1)$ matrix and one can show that if $\left|D\vec{P}(x_0)\right| < 1$, then the periodic orbit Γ is asymptotically stable.

The linearization of the system about Γ is defined as the non-autonomous linear system $\dot{x} = A(t)x, \quad [1]$

where $A(t) = D\vec{f}(\gamma(t))$ is a continuous T-periodic function.

Fundamental Matrix Solution: A matrix $\varphi(t)$ where all columns are linearly independent solutions of [1].

Principal Fundamental Matrix Solution: A fundamental matrix solution $\Phi(t)$ where there exists t_0 such that $\Phi(t_0)$ is the identity.

The matrix $D\vec{P}(x_0)$ is determinable by a fundamental matrix for the linearization of the system about the periodic orbit Γ.

Floquet theory is a branch of ODE theory regarding solutions to periodic linear differential equations of the form [1] with $A(t)$ a piecewise continuous periodic function with period T and through which the stability of solutions is revealed.

Floquet theory gives a canonical form for each fundamental matrix solution of [1]. The theory gives a coordinate change: $y = Q^{-1}(t)x$, with $Q(t + 2T) = Q(t)$ that transforms the periodic system to a traditional linear system with constant, real coefficients. Note that the solutions of the linear differential equation form a vector space. A principal fundamental matrix can be constructed from a fundamental matrix using $\Phi(t) = \varphi(t)\varphi^{-1}(t_0)$. The solution of the linear differential equation with the initial condition $x(0) = x_0$ is $x(t) = \varphi(t)\varphi^{-1}(0)x_0$ where $\varphi(t)$ is any fundamental matrix solution.

Theorem: For all $t \in \mathbb{R}$, $\varphi(t+T) = \varphi(t)\varphi^{-1}(0)\varphi(T)$. Here $\varphi^{-1}(0)\varphi(T)$ is known as the **monodromy matrix**. In addition, for each matrix B (possibly complex) such that: $e^{TB} = \varphi^{-1}(0)\varphi(T)$, there is a periodic (period T) matrix function $t \to P(t)$ such that:$\varphi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$.Also, there is a real matrix R and a real periodic (period- $2T$) matrix function $t \to Q(t)$ such that: $\varphi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$

If $\varphi(t)$ is a fundamental matrix for [1] which satisfies $\varphi(0) = I$, then $\left|D\vec{P}(x_0)\right| = |\varphi(T)|$ for any point $x_0 \in \Gamma$. It then follows from the above theorem that $\left|D\vec{P}(x_0)\right| = |e^{TB}|$. The eigenvalues of e^{BT} given by $e^{\lambda_j T}$ where $\lambda_j,\ j = 1, 2, \ldots, n$, are the eigenvalues of the matrix B. The eigenvalues of B, λ_j, are called **characteristic exponents** of $\gamma(t)$, and the eigenvalues of e^{BT}, $e^{\lambda_j T}$, are called the **characteristic multipliers** of $\gamma(t)$.

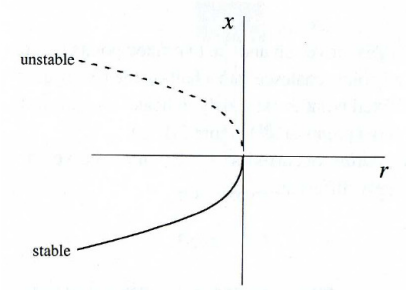
Local Bifurcations

Local bifurcations in a dynamical system occur when a parameter change (of an evolution function) causes the stability of an equilibrium (or fixed point) to change. They can be analysed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a **Floquet multiplier** with modulus equal to one. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighbourhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local').

Consider the continuous dynamical system described by the ODE: $\dot{x} = f(x, \lambda)$, with $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. A local bifurcation occurs at (x_0, λ_0) if the Jacobian matrix df_{x_0,λ_0}has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a **steady state bifurcation**, but if the eigenvalue is non-zero but purely imaginary, this is a **Hopf bifurcation**.

For discrete dynamical systems, consider the system: $x_{n+1} = f(x_n, \lambda)$. Then a local bifurcation occurs at (x_0, λ_0) if the matrix df_{x_0,λ_0} has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a **saddle-node** (often called **fold bifurcation** in maps), **transcritical** or **pitchfork** bifurcation. If the eigenvalue is equal to -1, it is a **period-doubling** (or **flip**) bifurcation, and otherwise, it is a **Hopf bifurcation** (an example here is the Neimark–Sacker (secondary Hopf) bifurcation). Let's elaborate on these:

Saddle-node (fold) Bifurcation



A local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other. The term "saddle-node bifurcation" is most often used in reference to continuous dynamical systems. In discrete dynamical systems, the same bifurcation is often instead called a fold bifurcation. Another name is blue skies bifurcation in reference to the sudden creation of two fixed points. If the phase space is one-dimensional, one of the equilibrium points is unstable (**the saddle**), while the other is stable (**the node**). Saddle-node bifurcations may be associated with **hysteresis loops** and **catastrophes**.

Normal form: A typical example of a differential equation with a saddle-node bifurcation is: $\frac{dx}{dt} = r + x^2$. Here x is the state variable and **r** is the bifurcation parameter.

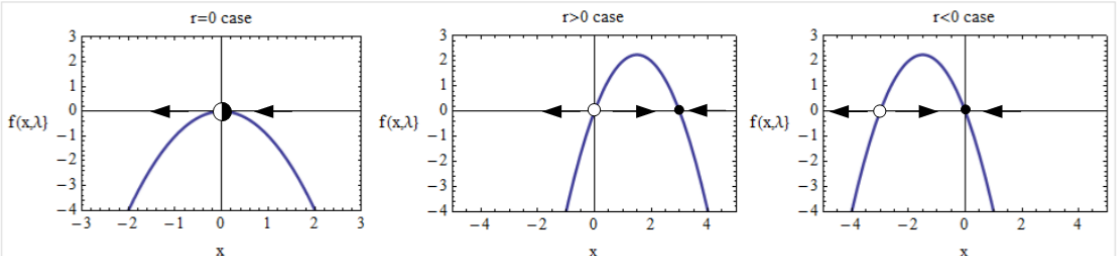
- If $r < 0$ there are two equilibrium points, a stable equilibrium point at $-\sqrt{-r}$ and an unstable one at $+\sqrt{-r}$.
- At $r = 0$ (the bifurcation point) there is exactly one equilibrium point. At this point the fixed point is no longer hyperbolic. In this case the fixed point is called a saddle-node fixed point.
- If $r > 0$ there are no equilibrium points.

Saddle node bifurcation: A scalar differential equation $\frac{dx}{dt} = f(r,x)$ which has a fixed point at $x = 0$ for $r = 0$ with $\frac{\partial f}{\partial x}(0,0) = 0,$ is locally topologically equivalent to $\frac{dx}{dt} = r \pm x^2$, provided it satisfies $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$ and $\frac{df}{dr}(0,0) \neq 0$. The first condition is the **nondegeneracy condition** and the second condition is the **transversality condition**.

Hysteresis is the dependence of the state of a system on its history. For a system with hysteresis, a **hysteresis loop** is the loop formed by the function when the independent variable is increased and decreased repeatedly over the part of the domain responsible for the hysteresis.

Catastrophe Theory: Analyzes degenerate critical points of the potential function — points where not just the first derivative, but one or more higher derivatives of potential function are also zero. These are called germs of the catastrophe geometries. The degeneracy of these critical points can be unfolded by expanding the potential function as Taylor series in small perturbations of the parameters. When degenerate points are not merely accidental, but structurally stable, they exist as organising centres for particular geometric structures of lower degeneracy, with critical features in the parameter space around them. If the potential function depends on two or fewer active variables, and four or fewer active parameters, then there are only seven generic structures for these bifurcation geometries, with corresponding standard forms into which the Taylor series around the catastrophe germs can be transformed by diffeomorphism (a smooth transformation whose inverse is also smooth).

Transcritical Bifurcation

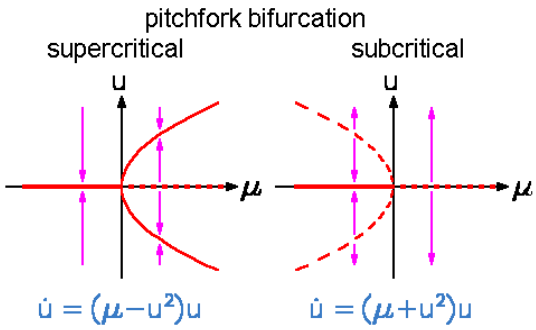


A fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied. In other words, both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide. So the unstable fixed point becomes stable and vice versa.

Normal form: $\frac{dx}{dt} = rx - x^2$. This equation is similar to the logistic equation but in this case we allow r and x to be positive or negative. The two fixed points are at $x = 0$ and $x = r$. When the parameter r is negative, the fixed point at $x = 0$ is stable and the fixed point $x = r$ is unstable. But for $r > 0$, the point at $x = 0$ is unstable and the point at $x = r$ is stable. So the bifurcation occurs at $r = 0$.

A typical example (in real life) could be the consumer-producer problem where the consumption is proportional to the (quantity of) resource. For example: $\frac{dx}{dt} = rx(1 - x) - px,$ where $rx(1 - x)$ is the logistic equation of resource growth; and px is the consumption, proportional to the resource x.

Pitchfork Bifurcation



Local bifurcation where the system transitions from one fixed point to three fixed points. Pitchfork bifurcations, like Hopf bifurcations have two types – supercritical and subcritical. In continuous dynamical systems described by ODEs—i.e. flows—pitchfork bifurcations occur generically in systems with symmetry.

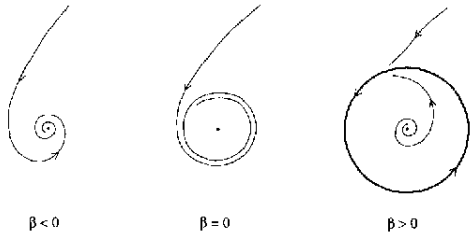
Supercritical Normal Form: $\frac{dx}{dt} = rx - x^3$. For negative values of r, there is one stable equilibrium at $x = 0$. For $r > 0$ there is an unstable equilibrium at $x = 0$, and two stable equilibria at $x = \pm\sqrt{r}$.

Subcritical Normal Form: $\frac{dx}{dt} = rx + x^3$. In this case, for $r < 0$ the equilibrium at $x = 0$ is stable, and there are two unstable equilibria at $x \pm \sqrt{-r}$. For $r > 0$ the equilibrium at $x = 0$ is unstable.

Formal definition: An ODE $\dot{x} = f(x,r)$, described by a one parameter function $f(x,r)$ with $r \in \mathbb{R}$ satisfying: $-f(x,r) = f(-x,r)$, (f is an odd function), where: $\frac{\partial f}{\partial x}(0,r_0) = 0$, $\frac{\partial^2 f}{\partial x^2}(0,r_0) = 0$, $\frac{\partial^3 f}{\partial x^3}(0,r_0) \neq 0$, $\frac{\partial f}{\partial r}(0,r_0) = 0$, and $\frac{\partial^2 f}{\partial r \partial x}(0,r_0) \neq 0$; has a pitchfork bifurcation at $(x,r) = (0,r_0)$. The form of the pitchfork is given by the sign of the third derivative: $\frac{\partial^3 f}{\partial x^3}(0,r_0)\begin{cases} < 0, \text{ supercritical,} \\ > 0, \text{ subcritical.} \end{cases}$

Note that subcritical and supercritical describe the stability of the outer lines of the pitchfork (dashed or solid, respectively) and are not dependent on which direction the pitchfork faces. For example, the negative of the first ODE above, $\dot{x} = x^3 - rx$, faces the same direction as the first picture but reverses the stability.

Hopf Bifurcation



A critical point where a system's stability switches and a periodic solution arises. More accurately, it is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues - of the linearization around the fixed point - crosses the complex plane's imaginary axis. Under reasonably generic assumptions about the dynamical system, a small-amplitude limit cycle branches from the fixed point. The limit cycle is orbitally stable if a specific quantity called the **first Lyapunov coefficient** is negative, and the bifurcation is supercritical. Otherwise it is unstable and the bifurcation is subcritical.

Normal form: $\frac{dz}{dt} = z((\lambda + i) + b|z|^2)$, where z, b are both complex and λ is a parameter.

Writing: $b = \alpha + i\beta$, the number α is called the **first Lyapunov coefficient**.

- **Supercritical**: If α is negative then there is a stable limit cycle for $\lambda > 0$:

  $z(t) = re^{i\omega t}$, where $r = \sqrt{\frac{-\lambda}{\alpha}}$, and $\omega = 1 + \beta r^2$.

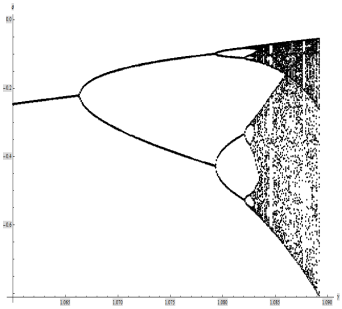
- **Subcritical**: If α is positive then there is an unstable limit cycle for $\lambda < 0$.

The following theorem works for steady points with one pair of conjugate nonzero purely imaginary eigenvalues. It tells

the conditions under which a bifurcation phenomenon occurs.

Theorem: Let J_0 be the Jacobian of a continuous parametric dynamical system evaluated at a steady point Z_e. Suppose that all eigenvalues of J_0 have negative real parts except one conjugate nonzero purely imaginary pair $\pm i\beta$. A Hopf bifurcation arises when these two eigenvalues leave the imaginary axis to gain a positive real part, because of a variation of the system parameters.

Period-Doubling (flip) Bifurcation



In a discrete dynamical system, a period-doubling bifurcation is one in which a slight change in a parameter value in the system's equations leads to the system switching to a new behavior with twice the period of the original system. With the doubled period, it takes twice as many iterations as before for the numerical values visited by the system to repeat themselves.

A **period doubling cascade** is a sequence of doublings and further doublings of the repeating period, as the parameter is adjusted further and further.

Period doubling bifurcations can also occur in continuous dynamical systems, namely when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one.

Example Logistic Map

Consider the following simple dynamics: $x_{n+1} = rx_n(1 - x_n)$ where x_n, lies in $[0, 1]$, and changes over time according to the parameter $r \in (0, 4]$. This classic example is a simplified version of the logistic map.

For μ between 1 and 3, x_n converges to the stable fixed point $x_* = \frac{r-1}{r}$. Then, for r between 3 and 3.44949, x_n converges to a permanent oscillation between two values x_* and x_*' that depend on r. As r grows larger, oscillations between 4 values, then 8, 16, 32, etc. appear. These period-doublings culminate at $r \approx 3.56995$ from where more complex regimes appear. As r increases, there are some intervals where most starting values will converge to one or a small number of stable oscillations, such as near $r = 3.83$.

In the interval where the period is 2^n for some positive integer n, not all the points actually have period n. These are single points, rather than intervals. These points are said to be in unstable orbits, since nearby points do not approach the same orbit as them.

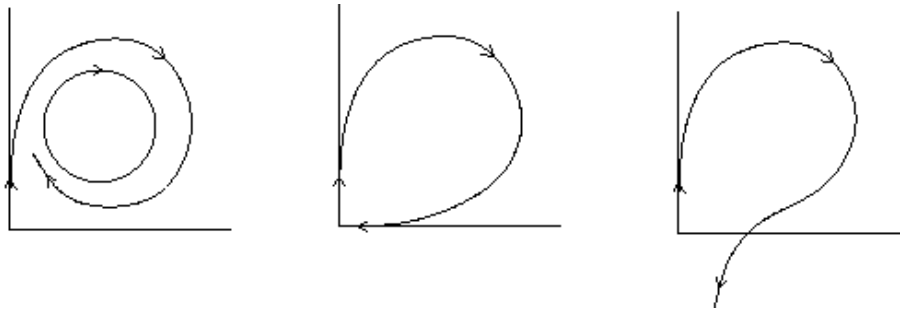
Global Bifurcations

Often occur when larger invariant sets of the system 'collide' with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria (fixed points). Examples of global bifurcations:

- **Homoclinic** bifurcation in which a limit cycle collides with a saddle point.
- **Heteroclinic** bifurcation in which a limit cycle collides with two or more saddle points.
- **Infinite-Period** bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.
- **Blue Sky Catastrophe** in which a limit cycle collides with a nonhyperbolic cycle.

Global bifurcations can also involve more complicated sets such as **chaotic attractors** (e.g. **crises**).

Homoclinic Bifurcation

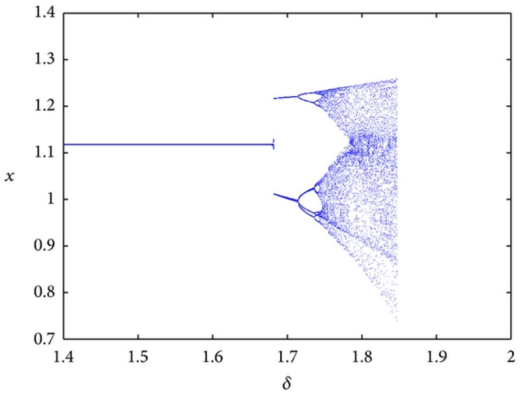


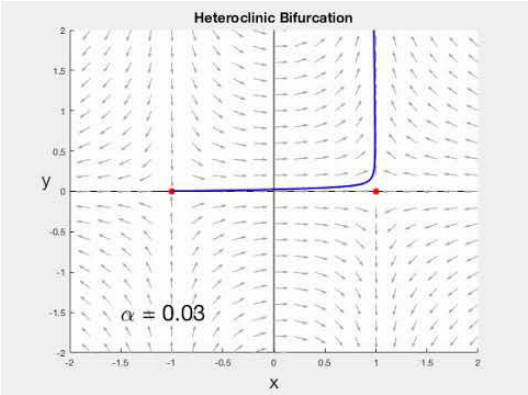
Occurs when a periodic orbit collides with a saddle point. **Left panel**: For small parameter values, there is a saddle point at the origin and a limit cycle in the first quadrant. **Middle panel**: As the bifurcation parameter increases, the limit cycle grows until it exactly intersects the saddle point, yielding an orbit of infinite duration. **Right panel**: When the bifurcation parameter increases further, the limit cycle disappears completely.

The image above shows a phase portrait before, at, and after a homoclinic bifurcation in 2D. The periodic orbit grows until it collides with the saddle point. At the bifurcation point the period of the periodic orbit has grown to infinity and it has become a homoclinic orbit. After the bifurcation there is no longer a periodic orbit.

Homoclinic bifurcations can occur supercritically or subcritically. The variant above is the "small" or "type I" homoclinic bifurcation. In 2D there is also the "big" or "type II" homoclinic bifurcation in which the homoclinic orbit "traps" the other ends of the unstable and stable manifolds of the saddle. In three or more dimensions, higher codimension bifurcations can occur, producing complicated, possibly chaotic dynamics.

Heteroclinic Bifurcation





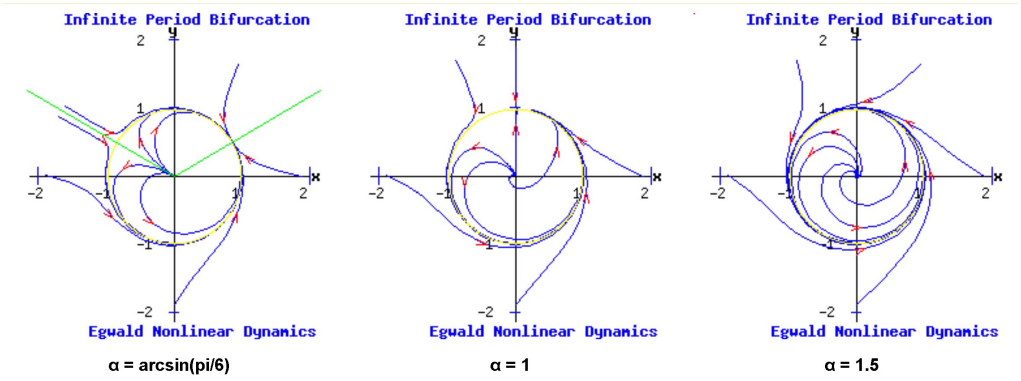
resonance **transverse**

Heteroclinic bifurcation is of two types: resonance bifurcations and transverse bifurcations. Both types of bifurcation will result in the change of stability of the heteroclinic cycle.

At a **resonance bifurcation**, the stability of the cycle changes when an algebraic condition on the eigenvalues of the equilibria in the cycle is satisfied. This is usually accompanied by the birth or death of a periodic orbit.

A **transverse bifurcation** of a heteroclinic cycle is caused when the real part of a transverse eigenvalue of one of the equilibria in the cycle passes through zero. This will also cause a change in stability of the heteroclinic cycle.

Infinite-Period Bifurcation

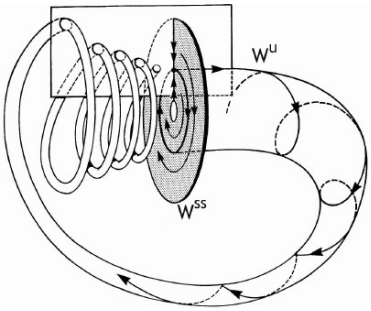


$$\frac{d\theta}{dt} = \alpha - \sin\theta$$

Occurs when two fixed points emerge on a limit cycle. As the limit of a parameter approaches a certain critical value, the speed of the oscillation slows down and the period approaches infinity. The infinite-period bifurcation occurs at this critical value. Beyond the critical value, the two fixed points emerge continuously from each other on the limit cycle to disrupt the oscillation and form two saddle points.

In the image above, a one dimensional nonuniform oscillator, exhibiting the properties of a type one saddle-node bifurcation. When $\alpha > 1$, counterclockwise flow varies with the value of θ, having a maximum at $\theta = -\frac{\pi}{2}$, and a minimum at $\theta = \frac{\pi}{2}$. When $\alpha = \alpha_c = 1$, the equation has a half-stable fixed point on the unit circle at $\theta = \frac{\pi}{2}$. For $\alpha < 1$, the fixed point splits into a stable fixed point and a second unstable fixed point on the unit circle.

Blue Sky Catastrophe



This type of bifurcation is characterised by both the period and length of the orbit approaching infinity as the control parameter approaches a finite bifurcation value, but with the orbit still remaining within a bounded part of the phase space, and without loss of stability before the bifurcation point. In other words, the orbit vanishes into the blue sky.

Above, we see Blue sky bifurcation in action: the unstable manifold W^u comes back to the saddle-node periodic orbit while making infinitely many revolutions in the stable, node, region separated from the saddle region by the strongly stable manifold W^{ss}. The strong transverse contraction transforms the homoclinic connection into a stable periodic orbit slowing down near the "phantom" of the saddle-node orbit.

Ergodic Systems

In many dynamical systems, it is possible to choose the coordinates of the system so that the volume (really a ν-dimensional volume) in phase space is invariant. This happens for **mechanical systems** derived from **Newton's laws** as long as the coordinates are the position and the momentum and the volume is measured in units of (position) × (momentum). The flow takes points of a subset A into the points $\Phi^t(A)$ and invariance of the phase space means that: $vol(A) = vol(\Phi^t(A))$.

In the **Hamiltonian formalism**, given a coordinate it is possible to derive the appropriate (generalized) momentum such

that the associated volume is preserved by the flow. The volume is said to be computed by the **Liouville Measure**: $\prod_i dq_i dp_i$.

In a **Hamiltonian System**, not all possible configurations of position and momentum can be reached from an initial condition. Because of energy conservation, only the states with the same energy as the initial condition are accessible. The states with the same energy form an **energy shell** Ω, a sub-manifold of the phase space. The volume of the energy shell, computed using the Liouville measure, is preserved under evolution.

For systems where the volume is preserved by the flow, Poincaré discovered the **recurrence theorem**: Assume the phase space has a finite Liouville volume and let F be a phase space volume-preserving map and A a subset of the phase space. Then almost every point of A returns to A infinitely often under F. The Poincaré recurrence theorem was used by Zermelo to object to Boltzmann's derivation of the increase in entropy in a dynamical system of colliding atoms.

One of the questions raised by Boltzmann's work was the possible equality between time averages and space averages, what he called the **ergodic hypothesis**. The hypothesis states that the portion of time a typical trajectory spends in a region A is $vol(A)/vol(\Omega)$.

The ergodic hypothesis turned out not to be the essential property needed for the development of **statistical mechanics**, and a series of other ergodic-like properties were introduced to capture the relevant aspects of physical systems. Koopman approached the study of ergodic systems by the use of functional analysis. An **observable** a is a function that to each point of the phase space associates a number (say instantaneous pressure, or average height). The value of an observable can be computed at another time by using the evolution function Φ^t. This introduces an operator U^t, the **transfer operator**: $(U^t a)(x) = a(\Phi^{-t}(x))$.

By studying the **spectral properties** of the linear operator U it becomes possible to classify the ergodic properties of Φ^t. In using the Koopman approach of considering the action of the flow on an observable function, the finite-dimensional nonlinear problem involving Φ^t gets mapped into an infinite-dimensional linear problem involving U.

The Liouville measure restricted to the energy surface Ω is the basis for the averages computed in **equilibrium statistical mechanics**. An average in time along a trajectory is equivalent to an average in space computed with the Boltzmann factor $e^{-\beta H}$.

Birkhoff Theorem: Let $T : X \to X$ be a measure-preserving transformation on a measure space (X, Σ, μ) and suppose f is a μ-integrable function, i.e. $f \in L^1(\mu)$. Then we define the following averages:

- **Time average**: This is defined as the average (if it exists) over iterations of T starting from some initial point x: $\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$.
- **Space average**: If $\mu(X)$ is finite and nonzero, we can consider the space or phase average of f: $\bar{f} = \frac{1}{\mu(X)} \int f d\mu$. (For a probability space, $\mu(X) = 1$.)

In general the time average and space average may be different. But if the transformation is ergodic, and the measure is invariant, then the time average is equal to the space average almost everywhere.

Von Neumann Theorem: Asserts the existence of a time average along each trajectory.

Mixing: Several different definitions for mixing exist, including strong mixing, weak mixing and topological mixing, with the last not requiring a measure to be defined. Some of the different definitions of mixing can be arranged in a hierarchical order; thus, strong mixing implies weak mixing. Furthermore, weak mixing (and thus also strong mixing) implies ergodicity: that is, every system that is weakly mixing is also ergodic (and so one says that mixing is a "stronger" notion than ergodicity).

Equidistribution: That the sequence $a, 2a, 3a, \ldots \mod 1$ is uniformly distributed on the circle $\mathbb{R}/\mathbb{Z}$, when a is an irrational number. It is a special case of the ergodic theorem where one takes the normalized angle measure $\mu = \frac{d\theta}{2\pi}$.

Nonlinear Dynamical Systems and Chaos

Simple nonlinear dynamical systems and even piecewise linear systems can exhibit a completely unpredictable behavior, which might seem to be random, despite the fact that they are fundamentally deterministic. This seemingly unpredictable

behavior has been called **chaos**.

Chaos: Although no universally accepted mathematical definition of chaos exists, a commonly used definition originally formulated says that, to classify a dynamical system as chaotic, it must have these properties:

- it must be sensitive to initial conditions,
- it must be topologically transitive,
- it must have dense periodic orbits.

In some cases, the last two properties in the above have been shown to actually imply sensitivity to initial conditions. In these cases, while it is often the most practically significant property, "sensitivity to initial conditions" need not be stated in the definition. If attention is restricted to intervals, the second property implies the other two. An alternative, and in general weaker, definition of chaos uses only the first two properties in the above list.

Hyperbolic Systems are precisely defined dynamical systems that exhibit the properties ascribed to chaotic systems. In hyperbolic systems the tangent space perpendicular to a trajectory can be well separated into two parts: one with the points that converge towards the orbit (the **stable manifold**) and another of the points that diverge from the orbit (the **unstable manifold**).

This branch of mathematics deals with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like "*Will the system settle down to a steady state in the long term, and if so, what are the possible attractors?*" or "*Does the long-term behavior of the system depend on its initial condition?*"

Note that the chaotic behavior of complex systems is not the issue. Meteorology has been known for years to involve complex — even chaotic — behavior. Chaos theory has been so surprising because chaos can be found within almost trivial systems. The **logistic map** is only a second-degree polynomial; the **horseshoe map** is piecewise linear.

Attractor

Let t represent time and let $f(t, \bullet)$ be a function which specifies the dynamics of the system. That is, if a is a point in an n-dimensional phase space, representing the initial state of the system, then $f(0, a) = a$ and, for a positive value of t, $f(t, a)$ is the result of the evolution of this state after t units of time. For example, if the system describes the evolution of a free particle in one dimension then the phase space is the plane $\mathbb{R}^2$ with coordinates (x, v), where x is the position of the particle, v is its velocity, $a = (x, v)$, and the evolution is given by: $f(t, (x, v)) = (x + tv, v)$.

An **attractor** is a subset A of the phase space characterized by the following three conditions:

- A is forward invariant under f: if a is an element of A then so is $f(t, a)$, for all $t > 0$.
- There exists a neighborhood of A, called the basin of attraction for A and denoted $B(A)$, which consists of all points b that "enter A in the limit $t \to \infty$". More formally, $B(A)$ is the set of all points b in the phase space with the following property:
  For any open neighborhood N of A, there is a positive constant T such that $f(t, b) \in N$ for all real $t > T$.
- There is no proper (non-empty) subset of A having the first two properties.

Since the basin of attraction contains an open set containing A, every point that is sufficiently close to A is attracted to A.

How do you solve a Differential Equation?

Solving differential equations is not like solving algebraic equations. Not only are their solutions often unclear, but whether solutions are unique or exist at all are also notable subjects of interest. There are very few methods of solving **nonlinear differential equations** exactly; those that are known typically depend on the equation having particular **symmetries**. Nonlinear differential equations can exhibit very complicated behavior over extended time intervals, characteristic of chaos. Only the simplest differential equations are solvable by explicit formulas; however, some **properties of solutions** of a given differential equation may be determined without finding their exact form. If a closed-form expression for the solution is not available, the solution may be **numerically approximated** using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

List of Methods

1st Order Differential Equations $y' + p(x)y = q$

Linear Homogeneous:

Separable Equations

Linear Nonhomogeneous:

Form $y' = q(x)$. Integrate!

Integrating Factor: Form $y' + p(x)y = q$. Then $\rho(x) = e^{\int P dx}$, $y = \frac{1}{\rho}\int p(x)q(x)dx$.

Nonlinear Homogeneous:

Separable: Form $N(y)y' = M(x)$. Homogeneous DEQ can be converted into a separable equation through a sufficient change of variables
Bernoulli: Form $y' + p(x)y = y^n$.

Nonlinear Nonhomogeneous:

Exact Equations: Form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, where there exists ψ with $\psi_x = M(x,y)$ and $\psi_y = N(x,y)$.
Also: **Series Solutions** or **Numerical Methods** (see below), notably: **Euler's Method** (approximation):
$y_{n+1} = y_n + \Delta x \cdot f(x_n, y_n),\ n \geq 0.$

2nd Order Differential Equations: $ay'' + by' + cy = q.$

Linear Homogeneous: Notable because they have solutions that can be added together in linear combinations to form further solutions.

Characteristic Polynomial, e^{rx}, and **Principle of Superposition**.

Linear Nonhomogeneous: Use: $y = y_c + y_p$

Undetermined Coefficients: Solve homogeneous y_c, generate trial solution y_t, plug this into the nonhomogeneous equation, and determine coefficients.
Variation of Parameters: $y_p = -y_1\int \frac{y_2 q}{W}dt + y_2\int \frac{y_1 q}{W}dt$, where $W(y_1, y_2)$ is the Wronskian.
Laplace Transform: More time intensive for simpler differential equations. But applicable to larger class of equations (forcing function can be more complicated). $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)$. Calculate: $a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{q\}$. Then solve resulting algebraic equation, and use inverse transform to get y.

Nonlinear: Series Solutions or Numerical Methods.

Systems of Differential Equations

Linear Homogeneous

Matrix Form: Identify the eigenvalues λ and eigenvectors v, and then form the solutions $ce^{\lambda x}\vec{v}$, add them in a linear combination.
Laplace Transforms

Linear Nonhomogeneous

Variation of Parameters
Undetermined Coefficients

Nonlinear: Series Solutions or Numerical Methods.

Series Solutions

Ordinary and Singular Points: Given $p(x)y'' + q(x)y' + r(x)y = 0,$ we say that $x = x_0$ is an ordinary point if both $\frac{q}{p}$ and $\frac{r}{p}$ are analytic at $x = x_0$. If p, q, r are polynomials, this is equivalent to $p(x_0) \neq 0$.

Series Solutions: Used to generate a solution around an ordinary point. Useful in solving (or at least getting an approximation of the solution to) DEQs with coefficients that are not constants. Assume the solution takes the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$ substitute this into the DEQ, and solve for a_n.

Euler DEQs: Of the form $ax^2y'' + bxy' + cy = 0.$ Illustrates how to get a solution to at least one type of DEQ at a singular point. Assuming $x > 0$ and the solution has form $y = x^r$, we plug this into the DEQ, solve the result algebraically for $r_i = a \pm ib$, and form the solution (depending upon multiplicity of roots) as $y = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x)$.

Numerical Methods.

Numerical methods for solving first-order IVPs often fall into one of two categories: **Single** and **Multistep** methods. A numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution.

Single-Step: Runge–Kutta Family of implicit and explicit iterative methods, which include the Euler Method (see above), used in temporal discretization for the approximate solutions of ODEs. Single-step methods (such as Euler's method) refer to only one previous point and its derivative to determine the current value. Methods such as Runge–Kutta take some intermediate steps (for example, a half-step) to obtain a higher order method, but then discard all previous information before taking a second step.

Linear multistep: Multistep methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it.

Picard–Lindelöf Theorem: existence and uniqueness theorem gives a set of conditions under which an initial value problem has a unique solution. Consider the initial value problem: $y'(t) = f(t, y(t))$, and $y(t_0) = y_0$.

Suppose f is uniformly Lipschitz continuous in y (meaning the Lipschitz constant can be taken independent of t) and continuous in t, then for some value $\varepsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

By integrating both sides, any function satisfying the differential equation must also satisfy the integral equation (Picard Integral): $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$.

– –

Implicit Function Theorem: A tool that given relation $f_i(x_1, \ldots, x_n, y_1, \ldots, y_m; \mu) = f_i(\mathbf{x}, \mathbf{y}; \mu) = 0$ and some initial point $(\mathbf{x}_0, \mathbf{y}_0)$ such that $f_i(\mathbf{x}_0, \mathbf{y}_0; \mu) = 0,$ tells of the existence of a function of several real variables $\mathbf{y} = g(\mathbf{x}; \mu)$ which satisfies the relation. It does so by representing the relation as the graph of a function. There may not be a single function whose graph can represent the entire relation, but there may be such a function on a restriction of the domain of the relation.

More precisely, given a system of m equations $f_i(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0, \quad i = 1, \ldots, m$ (often abbreviated into $F(x, y) = 0$), the theorem states that, under a mild condition on the partial derivatives (with respect to the y_is) at a point, the m variables y_i are differentiable functions of the x_j in some neighborhood of the point. As these functions can generally not be expressed in closed form, they are implicitly defined by the equations, and this motivated the name of the theorem. In other words, under a mild condition on the partial derivatives, the set of zeros of a system of equations is locally the graph of a function.

First example

If we define the function $f(x, y) = x^2 + y^2$, then the equation $f(x, y) = 1$ cuts out the unit circle as the level set $\{(x, y) | f(x, y) = 1\}$. There is no way to represent the unit circle as the graph of a function of one variable $y = g(x)$ because for each choice of $x \in (-1, 1)$, there are two choices of y, namely $\pm\sqrt{1 - x^2}$.

However, it is possible to represent part of the circle as the graph of a function of one variable. If we let $g_1(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, then the graph of $y = g_1(x)$ provides the upper half of the circle. Similarly, if $y = -g_1(x)$, then the graph of $y = g_2(x)$ gives the lower half of the circle.

The purpose of the implicit function theorem is to tell us the existence of functions like $g_1(x)$ and $g_2(x)$, even in

situations where we cannot write down explicit formulas. It guarantees that $g_1(x)$ and $g_2(x)$ are differentiable, and it even works in situations where we do not have a formula for $f(x,y)$.

Definitions:

Let $f : R^{n+m} \to R^m$ be a continuously differentiable function. We think of R^{n+m} as the Cartesian product $R^n \times R^m$, and we write a point of this product as $(x,y) = (x_1,\ldots,x_n,y_1,\ldots,y_m)$. Starting from the given function f, our goal is to construct a function $g : R^n \to R^m$ whose graph $(x,g(x))$ is precisely the set of all (x,y) such that $f(x,y) = 0$.

As noted above, this may not always be possible. We will therefore fix a point $(a,b) = (a_1,\ldots,a_n,b_1,\ldots,b_m)$ which satisfies $f(a,b) = 0$, and we will ask for a g that works near the point (a,b). In other words, we want an open set U of R^n containing a, an open set V of R^m containing b, and a function $g : U \to V$ such that the graph of g satisfies the relation f = 0 on $U \times V$, and that no other points within $U \times V$ do so. In symbols,

$$\big\{(x,g(x)) \mid x \in U\big\} = \big\{(x,y) \in U \times V \mid f(x,y) = 0\big\}.$$

To state the implicit function theorem, we need the Jacobian matrix of f, which is the matrix of the partial derivatives of f. Abbreviating $(a_1,\ldots,a_n,b_1,\ldots,b_m)$ to (a,b), the Jacobian matrix is

$$(Df)(a,b) = \left[\begin{array}{ccc|ccc} \partial_{x_1} f_1(a,b) & \ldots & \partial_{x_n} f_1(a,b) & \partial_{y_1} f_1(a,b) & \ldots & \partial_{y_m} f_1(a,b) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(a,b) & \ldots & \partial_{x_n} f_m(a,b) & \partial_{y_1} f_m(a,b) & \ldots & \partial_{y_m} f_m(a,b) \end{array}\right] = [X|Y].$$ The implicit function theorem says that if Y is an invertible matrix, then there are U, V, and g as desired. Writing all the hypotheses together gives the following statement.

Statement of the Theorem

Let $f : R^{n+m} \to R^m$ be a continuously differentiable function, and let R^{n+m} have coordinates (x,y). Fix a point $(a,b) = (a_1,\ldots,a_n,b_1,\ldots,b_m)$ with $f(a,b) = 0$. If the Jacobian matrix $J_{f,y}(a,b) = [(\partial f_i/\partial y_j)(a,b)]$ (this is the right-hand panel of the Jacobian matrix shown in the previous section) is invertible, then there exists an open set U of R^n containing a such that there exists a unique continuously differentiable function $g : U \to R^m$ such that: $g(a) = b$ and $f(x,g(x)) = 0$ for all $x \in U$. Moreover, the partial derivatives of g in U are given by the matrix product:
$\partial_{x_j} g(x) = -[J_{f,y}(x,g(x))]^{-1}_{m\times m}[\partial_{x_j} f(x,g(x))]_{m\times 1}$.

Lyapunov–Schmidt Reduction

Used to study solutions to nonlinear equations in the case when the implicit function theorem does not work.

Problem Set up

Let $f(x,\lambda) = 0$ be the given nonlinear equation, X,Λ, and Y are Banach spaces (Λ is parameter space). $f(x,\lambda)$ is C^p-map from a neighborhd of some point $(x_0,\lambda_0) \in X \times \Lambda$ to Y and the equation is satisfied at this point (x_0,λ_0). For case when the linear operator $f_x(x,\lambda)$ is invertible, IFT assures that there exists a solution $x(\lambda)$ satisfying the equation $f(x(\lambda),\lambda) = 0$ at least locally close to λ_0. In the opposite case, when the linear operator $f_x(x,\lambda)$ is non-invertible, the Lyapunov–Schmidt reduction can be applied in the following way.

Assumptions: One assumes that the operator $f_x(x,\lambda)$ is a Fredholm operator, that is: $\ker(f_x(x_0,\lambda_0)) =: X_1$ where X_1 has finite dimension, $range(f_x(x_0,\lambda_0)) =: Y_1$ has finite co-dimension and is a closed subspace in Y. Without loss of generality, one can also assume that $(x_0,\lambda_0) = (0,0)$.

Lyapunov–Schmidt construction

Let us split Y into the direct product $Y = Y_1 \oplus Y_2$, where $\dim Y_2 < \infty$. Let Q be the projection operator onto Y_1. Consider also the direct product $X = X_1 \oplus X_2$. Applying the operators Q and $I - Q$ to the original equation, one obtains the equivalent system: $Qf(x,\lambda) = 0$, $(I-Q)f(x,\lambda) = 0$. Let $x_1 \in X_1$ and $x_2 \in X_2$, then the first equation: $Qf(x_1 + x_2,\lambda) = 0$, can be solved with respect to x_2 by applying the **implicit function theorem** to the operator: $Qf(x_1 + x_2,\lambda) : X_2 \times (X_1 \times \Lambda) \to Y_1$ (now the conditions of the implicit function theorem are fulfilled). Thus, there exists a unique solution $x_2(x_1,\lambda)$ satisfying: $Qf(x_1 + x_2(x_1,\lambda),\lambda) = 0$.

However, implicit function theorem doesn't tell you what x_2 is. If the correct projections are chosen, some qualitative properties of x_2 (such as continuity and differentiability) can be obtained such that a Taylor expansion can be used to approximate x_2 (and subsequently $G(x_1)$) for use in the following steps. Now substituting $x_2(x_1, \lambda)$ into the second equation, one obtains the final finite-dimensional equation: $G(x_1) := (I - Q)f(x_1 + x_2(x_1, \lambda), \lambda) = 0$.

Indeed, this last equation is now finite-dimensional, since the range of $(I - Q)$ is finite-dimensional. This equation is now to be solved with respect to x_1, which is finite-dimensional, and parameters : λ.

Newton Polygon

A method for taking an implicit multivariate equation $f(u;\mu) = 0,$ and determining the branches of solutions near a given solution (u, μ). Consider for example the equation: $f(u;\mu) := \mu^4 + u\mu + u^3 + u\mu^2 = 0.$ (8.16)

At $\mu = 0$, we have a triple solution (counting with complex multiplicity) $u = 0$ at the origin (note that if our solution is not at the origin, we can change variables such that it is), and we ask how these three solutions unfold. Newton's strategy is to plot exponents in the Taylor expansion of f on the positive lattice (called a Newton's polygon); see Figure 8.5.

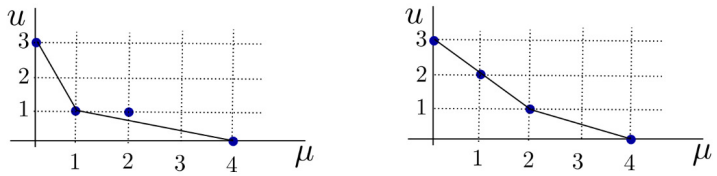


Figure 8.5: Exponents for (8.16) (left) and (8.17) (right) with associated Newton polygons as lower convex envelopes

One then draws the lower convex envelope, which is referred to as the Newton polygon. A line segment typically connect two exponents (which we call our **Leading Order Segment**), while all other exponents lie above/to the right of this particular line segment. Although, sometimes more exponents can lie on the same line segment, while still "most" others lie above it, as seen in the right panel of Figure 8.5 associated with (8.17). Each line segment gives a particular **scaling** that will make the terms associated with the powers on the line segment dominant terms and terms above the line segment higher-order terms.

Recall: $f(u;\mu) := \mu^4 + u\mu + u^3 + u\mu^2 = 0$

In our case of (8.16), we find two line segments. The Leading Order Segment is associated with the terms μu, u^3 and therefore suggests:

- Scaling: $\mu \sim u^2$ (near $(0,0)$) which we can accomplish by:
- Setting: $u = u_1\varepsilon$ and $\mu = \varepsilon^2$. (1)

Which yields from 8.16: $\varepsilon^8 + \varepsilon^3(u_1 + u_1^3) + \varepsilon^4 u_1^2 = 0$. Dividing by ε^3 $(\varepsilon^5 + (u_1 + u_1^3) + \varepsilon u_1^2)$ and subsequently setting $\varepsilon = 0$, we find $u_1 + u_1^3 = 0$ and three solutions $u_1 = 0, \pm i$. From (1), we have $u = u_1\varepsilon = u_1(\mu^{1/2}) = \pm i\mu^{1/2} + O(|\mu|)$.

We can now continue with the IFT (if those solutions were degenerate and the IFT not applicable, we would simply apply this Newton polygon procedure to the scaled equation) to find for the non-trivial solutions
$u = \pm i\mu^{1/2} + \frac{1}{2}\mu + O(|\mu|^{3/2})$.

The 2nd Order Segment of the Newton polygon gives us the expansion for the trivial solution $u_1 = 0$ as follows. We equate terms $u\mu$ and μ^4, which gives:

- Scaling: $u \sim \mu^3$ (near $(0,0)$) which we can accomplish by
- Setting: $u = u_2\varepsilon^3$ and $\mu = \varepsilon$. (2)
- Which yields from 8.16: $\varepsilon^4 + \varepsilon^4 u_2 + \varepsilon^9 u_2^3 + \varepsilon^7 u_2^2 = 0$.

Dividing by ε^4 $(1 + u_2 + \varepsilon^5 u_2^3 + \varepsilon^3 u_2^2)$ and subsequently setting $\varepsilon = 0$, we find $u_2 = -1$. From (2), we have $u = u_2\varepsilon^3 = u_2\mu^3 = -\mu^3 + O(\mu^4)$. Expanding further, $u = -\mu^3 - \mu^6$.

The case when several exponents lie on the same edge is illustrated by:

$f(u;\mu) := u^3 + u\mu^2 + 2\mu u^2 + \mu^4 = 0.$ (see graph above) (8.17)

Here, the Leading Order Segment gives:

- Scaling: $\mu \sim u$ (near $(0,0)$?) which we can accomplish by:
- Setting: $u = \varepsilon u_1$ and $\mu = \varepsilon$. Which yields from 8.17: $\varepsilon^3 u_1^3 + \varepsilon^3 u_1 + 2\varepsilon^3 u_1^2 + \varepsilon^4 = 0$.

Dividing by $\varepsilon^3 = 0$ and subsequently setting $\varepsilon = 0$, we find $u_1^3 + 2u_1^2 + u_1 = 0 \quad \Rightarrow \quad u_1 = 0$ or $u_1^2 + 2u_1 + 1 = 0 \quad \Rightarrow \quad u_1 = -1$ with multiplicity 2.

We therefore iterate our procedure. In order to get a **2nd order** correction v, we set $u_1 = -1 + v$ in 8.17b giving $(v-1)^3 + 2(v-1)^2 + (v-1) + \mu = v^3 - v^2 + \mu$, using the **second order** segment of the Newton polygon we have, $-v^2 \sim \mu \quad \Rightarrow \quad v = i\mu^{1/2}$. So, $u = u_1\varepsilon = \varepsilon(-1+v) = \varepsilon(-1 + i\mu^{1/2})$, and gathering all the calculated solutions: $u \in \left\{0,\ -\varepsilon,\ \varepsilon(-1 + i\mu^{1/2})\right\}$.

Inspecting the algorithm, one easily convinces one self that it yields always the correct number of solution branches, corresponding to the lowest pure power of u (if all terms possess a factor μ, one simply divides by this prefactor). A general theorem states that for analytic function f, this procedure always terminates and finds all solutions in terms of power series in some root $\mu^{1/q}$.

Wiki:

A tool for understanding the behaviour of polynomials over local fields. The Newton polygon is an effective device for understanding the leading terms ax^r of the power series expansion solutions to equations $P(F(x)) = 0$, where P is a polynomial with coefficients in $K[x]$, the polynomial ring; that is, implicitly defined algebraic functions. The exponents r here are certain rational numbers, depending on the branch chosen; and the solutions themselves are power series in $K[[y]]$ with $y = x^{1/d}$ for a denominator d corresponding to the branch. The Newton polygon gives an effective, algorithmic approach to calculating d.

Definition

A priori, given a polynomial over a field, the behaviour of the roots (assuming it has roots) will be unknown. Newton polygons provide one technique for the study of the behaviour of the roots.

Let K be a local field with discrete valuation v_K and let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ with $a_0 a_n \neq 0$. Then the Newton polygon of f is defined to be the lower convex hull of the set of points $P_i = (i, v_K(a_i))$, ignoring the points with $a_i = 0$. Restated geometrically, plot all of these points P_i on the xy-plane. Let's assume that the points indices increase from left to right (P_0 is the leftmost point, P_n is the rightmost point). Then, starting at P_0, draw a ray straight down parallel with the y-axis, and rotate this ray counter-clockwise until it hits the point P_{k_1} (not necessarily P_1). Break the ray here. Now draw a second ray from P_{k_1} straight down parallel with the y-axis, and rotate this ray counter-clockwise until it hits the point P_{k_2}. Continue until the process reaches the point Pn; the resulting polygon (containing the points $P_0, P_{k_1}, P_{k_2}, \ldots, P_{k_m}, P_n$) is the Newton polygon.

Another, perhaps more intuitive way to view this process is this : consider a rubber band surrounding all the points $P_0, \ldots, P_n$. Stretch the band upwards, such that the band is stuck on its lower side by some of the points (the points act like nails, partially hammered into the xy plane). The vertices of the Newton polygon are exactly those points.

Random Stuff

Strange Attractor: An attracting set that has zero measure in the embedding phase space and has fractal dimension. Trajectories within a strange attractor appear to skip around randomly. An attractor is called strange if it has a fractal structure. This is often the case when the dynamics on it are chaotic, but strange nonchaotic attractors also exist. If a strange attractor is chaotic, exhibiting sensitive dependence on initial conditions, then any two arbitrarily close alternative initial points on the attractor, after any of various numbers of iterations, will lead to points that are arbitrarily far apart (subject to the confines of the attractor), and after any of various other numbers of iterations will lead to points that are arbitrarily close together. Thus a dynamic system with a chaotic attractor is locally unstable yet globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the

attractor.

Morse Smale Systems: Smooth dynamical system whose non-wandering set consists of finitely many hyperbolic equilibrium points and hyperbolic periodic orbits and satisfying a transversality condition on the stable and unstable manifolds. Morse–Smale systems are structurally stable

Limit Sets

ω **Limit Point**: A point $\vec{y}$ is an ω-limit point of $\vec{x}$ for f provided there exists a sequence of n_k going to infinity as k goes to infinity such that $\lim_{k\to\infty} d(f^{n_k}(x), \vec{y}) = 0$.

ω-**Limit Theorem**: Let $f : X \to X$ be a continuous map on a complete metric space X.

Positively (**Negatively**) **Invariant Set** $\mathcal{K}$: With initial point $\vec{x}$, let $\mathcal{K} := \left\{\vec{y} \in \mathbb{R}^n | \varphi(\vec{y}) = 0 \text{ for some real valued function } \varphi\right\}$. The set is said to be positively (negatively) invariant if $\vec{x} \in \mathcal{K}$ implies that $f^n(\vec{x}) \in \mathcal{K}\ \forall n \geq 0$ (resp. $n \leq 0$).

Invariant Set: *a* set which is both positively and negatively invariant.

Forward (**Backward**) **Orbit** $O^{\pm}(\vec{x})$: $O^{\pm}(\vec{x}) := \left\{f^i : i \in \mathbb{Z}, i \geq 0 \text{ (resp. } i \leq 0)\right\}$.

- For any $\vec{x}$, $\omega(\vec{x}) = \cap_{N\geq 0}\, cl(\cup_{n\geq N} \{f^n(\vec{x})\})$. If f is invertible, then $\alpha(\vec{x}) = \cap_{N\leq 0}\, cl(\cup_{n\leq N} \{f^n(\vec{x})\})$.
- If $f^j(\vec{x}) = \vec{y}$ for some integer j, then $\omega(\vec{x}) = \omega(\vec{y})$. Also, $\alpha(\vec{x}) = \alpha(\vec{y})$ if f is invertible.
- For any $\vec{x}$, $\omega(\vec{x})$ is closed and positively invariant.
- If **i)** $\boldsymbol{O}^+(\vec{x})$ is contained in some compact subset of X or **ii)** f is 1-to-1, then $\omega(\vec{x})$ is invariant.
- Similarly, if f is invertible, then $\alpha(\vec{x})$ is closed and invariant.
- If $\boldsymbol{O}^+(\vec{x})$ is contained in compact subset of X (e.g., forward orbit is bounded in some Eucl. space), then $\omega(\vec{x})$ is nonempty & compact, and distance $d(f^n(\vec{x}), \omega(\vec{x}))$ goes to zero as n goes to infinity. Similarly, if $\boldsymbol{O}^-(\vec{x})$ is contained in compact subset of X, then $\alpha(\vec{x})$ is nonempty & compact, & $d(f^n(\vec{x}), \alpha(\vec{x}))$ goes to zero as n goes to minus infinity.
- If $D \subset X$ is closed & positively invariant, & $\vec{x} \in D$, then $\omega(\vec{x}) \subset D$. Similarly, if f is invertible & D is negatively invariant and $\vec{x} \in D$, then $\alpha(\vec{x}) \subset D$.
- If $\vec{y} \in \omega(\vec{x})$, then $\omega(\vec{y}) \subset \omega(\vec{x})$, and (if f is invertible) then $\alpha(\vec{y}) \subset \omega(\vec{x})$. Similarly, if f is invertible & $\vec{y} \in \alpha(\vec{x})$, then $\alpha(\vec{y}) \subset \alpha(\vec{x})$ and $\omega(\vec{y}) \subset \alpha(\vec{x})$.

Minimal Set: A set S is a minimal set for f provided i) S is a closed, nonempty, invariant set and ii) if B is a closed, nonempty, invariant set of S, then $B = S$. Clearly, any periodic orbit is a minimal set.

Nowhere Dense: A set S where the interior of the closure of S is the empty set, $\text{int}(cl(S)) = \emptyset$.

Totally Disconnected: S where the connected components are single points. In the real line, a closed set is nowhere dense if and only if it is totally disconnected. However, in the plane, a curve is nowhere dense but is not totally disconnected.

Perfect Set: S such that it is closed and every point $p \in S$ is the limit of points $q_n \in S$ with $q_n \neq p$.

Celestial Mechanics

History

Modern analytic celestial mechanics started with Isaac Newton's Principia of 1687. The name "celestial mechanics" is more recent than that. Newton wrote that the field should be called "rational mechanics." The term "dynamics" came in a little later with Gottfried Leibniz, and over a century after Newton, Pierre-Simon Laplace introduced the term "celestial mechanics." Prior to Kepler there was little connection between exact, quantitative prediction of planetary positions, using geometrical or arithmetical techniques, and contemporary discussions of the physical causes of the planets' motion.

Johannes Kepler

Johannes Kepler (1571–1630) was the first to closely integrate the predictive geometrical astronomy, which had been dominant from Ptolemy in the 2nd century to Copernicus, with physical concepts to produce a New Astronomy, Based upon Causes, or Celestial Physics in 1609. His work led to the modern laws of planetary orbits, which he developed using his physical principles and the planetary observations made by Tycho Brahe. Kepler's model greatly improved the accuracy of predictions of planetary motion, years before Isaac Newton developed his law of gravitation in 1686.

Isaac Newton

Isaac Newton (1642–1727) is credited with introducing the idea that the motion of objects in the heavens, such as planets, the Sun, and the Moon, and the motion of objects on the ground, like cannon balls and falling apples, could be described by the same set of physical laws. In this sense he unified celestial and terrestrial dynamics. Using Newton's law of universal gravitation, proving Kepler's Laws for the case of a circular orbit is simple. Elliptical orbits involve more complex calculations, which Newton included in his Principia.

Joseph-Louis Lagrange

After Newton, Lagrange (1736–1813) attempted to solve the three-body problem, analyzed the stability of planetary orbits, and discovered the existence of the Lagrangian points. Lagrange also reformulated the principles of classical mechanics, emphasizing energy more than force and developing a method to use a single polar coordinate equation to describe any orbit, even those that are parabolic and hyperbolic. This is useful for calculating the behaviour of planets and comets and such. More recently, it has also become useful to calculate spacecraft trajectories.

Simon Newcomb

Simon Newcomb (1835–1909) was a Canadian-American astronomer who revised Peter Andreas Hansen's table of lunar positions. In 1877, assisted by George William Hill, he recalculated all the major astronomical constants. After 1884, he conceived with A. M. W. Downing a plan to resolve much international confusion on the subject. By the time he attended a standardisation conference in Paris, France in May 1886, the international consensus was that all ephemerides should be based on Newcomb's calculations. A further conference as late as 1950 confirmed Newcomb's constants as the international standard.

Albert Einstein

Albert Einstein (1879–1955) explained the anomalous precession of Mercury's perihelion in his 1916 paper The Foundation of the General Theory of Relativity. This led astronomers to recognize that Newtonian mechanics did not provide the highest accuracy. Binary pulsars have been observed, the first in 1974, whose orbits not only require the use of General Relativity for their explanation, but whose evolution proves the existence of gravitational radiation, a discovery that led to the 1993 Nobel Physics Prize.

Classical Mechanics

Describes the motion of macroscopic objects, from projectiles to parts of machinery, and astronomical objects, such as spacecraft, planets, stars and galaxies. If the present state of an object is known it is possible to predict by the laws of classical mechanics how it will move in the future (determinism) and how it has moved in the past (reversibility).

The earliest development of classical mechanics is often referred to as Newtonian mechanics. It consists of the physical concepts employed by and the mathematical methods invented by Isaac Newton ($F = ma$) and Gottfried Wilhelm Leibniz and others in the 17th century to describe the motion of bodies under the influence of a system of forces.

Classical mechanics provides extremely accurate results when studying large objects that are not extremely massive and speeds not approaching the speed of light. When the objects being examined have about the size of an atom diameter, it becomes necessary to introduce the other major sub-field of mechanics: **quantum mechanics**. To describe velocities that are not small compared to the speed of light, **special relativity** is needed. In case that objects become extremely massive, **general relativity** becomes applicable. However, a number of modern sources do include relativistic mechanics into classical physics, which in their view represents classical mechanics in its most developed and accurate form.

Lagrangian Mechanics

A reformulation of classical mechanics, introduced by the Italian-French mathematician and astronomer Joseph-Louis Lagrange in 1788.

In Lagrangian mechanics, the trajectory of a system of particles is derived by solving the Lagrange equations in one of two forms: either the Lagrange equations of the first kind, which treat constraints explicitly as extra equations, often using Lagrange multipliers; or the Lagrange equations of the second kind, which incorporate the constraints directly by judicious choice of generalized coordinates. In each case, a mathematical function called the Lagrangian is a function of the generalized coordinates, their time derivatives, and time, and contains the information about the dynamics of the system.

No new physics are necessarily introduced in applying Lagrangian mechanics compared to Newtonian mechanics. It is, however, more mathematically sophisticated and systematic. Newton's laws can include non-conservative forces like friction; however, they must include constraint forces explicitly and are best suited to Cartesian coordinates. Lagrangian mechanics is ideal for systems with conservative forces and for bypassing constraint forces in any coordinate system. Dissipative and driven forces can be accounted for by splitting the external forces into a sum of potential and non-potential forces, leading to a set of modified Euler–Lagrange (EL) equations. Generalized coordinates can be chosen by convenience, to exploit symmetries in the system or the geometry of the constraints, which may simplify solving for the motion of the system. Lagrangian mechanics also reveals conserved quantities and their symmetries in a direct way, as a special case of Noether's theorem.

Intuitive Derivation of the Euler Lagrange Equations

https://farside.ph.utexas.edu/teaching/336L/Fluid/node266.html

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Hamiltonian Mechanics:

Hamiltonian mechanics is a theory developed as a reformulation of classical mechanics and predicts the same outcomes as non-Hamiltonian classical mechanics. It uses a different mathematical formalism, providing a more abstract understanding of the theory. Historically, it was an important reformulation of classical mechanics, which later contributed to the formulation of statistical mechanics and quantum mechanics. Hamiltonian mechanics was first formulated by William Rowan Hamilton in 1833, starting from Lagrangian mechanics, a previous reformulation of classical mechanics introduced by Joseph Louis Lagrange in 1788. In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates $r = (q,p)$, where each component of the coordinate q_i, p_i is indexed to the frame of reference of the system.

The time evolution of the system is uniquely defined by Hamilton's equations: $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$, $\frac{dq}{dt} = +\frac{\partial H}{\partial p}$, where $H = H(q,p,t)$ is the Hamiltonian, which often corresponds to the total energy of the system. For a closed system, it is the sum of the kinetic and potential energy in the system.

In Newtonian mechanics, the time evolution is obtained by computing the total force being exerted on each particle of the system, and from Newton's second law, the time-evolutions of both position and velocity are computed. In contrast, in Hamiltonian mechanics, the time evolution is obtained by computing the Hamiltonian of the system in the generalized coordinates and inserting it in the Hamilton's equations. This approach is equivalent to the one used in Lagrangian mechanics. In fact, the Hamiltonian is the Legendre transform of the Lagrangian when holding q and t fixed and defining p as the dual variable, and thus both approaches give the same equations for the same generalized momentum. The main motivation to use Hamiltonian mechanics instead of Lagrangian mechanics comes from the symplectic structure of Hamiltonian systems.

While Hamiltonian mechanics can be used to describe simple systems such as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time, its strength is shown in more complex dynamic systems, such as planetary orbits in celestial mechanics. The more degrees of freedom the system has, the more complicated its time evolution is and, in most cases, it becomes chaotic.

Hamiltonian System:

A Hamiltonian system is a dynamical system governed by Hamilton's equations. In physics, this dynamical system describes the evolution of a physical system such as a planetary system or an electron in an electromagnetic field. These systems can be studied in both Hamiltonian mechanics and dynamical systems theory.

Informally, a Hamiltonian system is a mathematical formalism developed by Hamilton to describe the evolution equations of a physical system. The advantage of this description is that it gives important insight about the dynamics, even if the initial value problem cannot be solved analytically. One example is the planetary movement of three bodies: even if there is no simple solution to the general problem, Poincaré showed for the first time that it exhibits deterministic chaos.

Formally, a Hamiltonian system is a dynamical system completely described by the scalar function $H(q,p,t)$, the Hamiltonian. The state of the system, r, is described by the generalized coordinates "momentum" p and "position" q where both p and q are vectors with the same dimension N. So, the system is completely described by the $2N$ dimensional vector $r = (q,p)$, and the evolution equation is given by the Hamilton's equations: $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$ and $\frac{dq}{dt} = +\frac{\partial H}{\partial p}$.

The trajectory $r(t)$ is the solution of the initial value problem defined by the Hamilton's equations and the initial condition $r(0) = r_0 \in \mathbb{R}^{2N}$.

Integral of Motion:

A **constant of motion** may be defined in a given force field as any function of phase-space coordinates (position and velocity, or position and momentum) **and time** that is constant throughout a trajectory. A subset of the constants of motion are the **integrals of motion**, or **first integrals**, defined as any functions of only the phase-space coordinates that are constant along an orbit. Every integral of motion is a constant of motion, but the converse is not true because a constant of motion may depend on time. Examples of integrals of motion are the angular momentum vector, $L = x \cdot v$, or

a Hamiltonian without time dependence, such as $H(x,v) = \frac{1}{2}v^2 + \Phi(x)$. An example of a function that is a constant of motion but not an integral of motion would be the function $C(x,v,t) = x - vt$ for an object moving at a constant speed in one dimension.

Symplectic Matrix:

A symplectic matrix is a $2n \times 2n$ matrix M with real entries that satisfies the condition: $M^T \Omega M = \Omega$, where Ω is a fixed $2n \times 2n$ nonsingular, skew-symmetric matrix. This definition can be extended to $2n \times 2n$ matrices with entries in other fields, such as the complex numbers.

Typically Ω is chosen to be the block matrix: $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. The matrix Ω has determinant +1 and has an inverse given by $\Omega - 1 = \Omega^T = -\Omega$.

Every symplectic matrix has determinant 1, and the $2n \times 2n$ symplectic matrices with real entries form a subgroup $Sp(2n, \mathbb{R})$ of the special linear group $SL(2n, \mathbb{R})$ (set of $n \times n$ matrices with determinant 1). Topologically, the symplectic group $Sp(2n, \mathbb{R})$ is a connected noncompact real Lie group of real dimension $n(2n + 1)$. The symplectic group can be defined as the set of linear transformations that preserve the symplectic form of a real symplectic vector space. Examples of symplectic matrices include the identity matrix and the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The Kepler Problem (Existence and Uniqueness):

In classical mechanics, the Kepler problem (formulated **after** Kepler) is a special case of the two-body problem. It starts by assuming two bodies interact by a central force F, that varies in strength as the inverse square of distance r between them (via Newton). The force may be attractive or repulsive. The "problem" to be solved is to find the position or speed of the two bodies over time given their masses and initial positions and velocities. Using classical mechanics, the solution can be expressed as a Kepler orbit using six orbital elements.

The Kepler problem is named after Johannes Kepler, who proposed **Kepler's laws of planetary motion** (which are part of classical mechanics and solve the problem for the orbits of the planets) and investigated the types of forces that would result in orbits obeying those laws (called **Kepler's inverse problem**).

The Kepler problem in gen. rel. produces more accurate predictions, especially in strong gravitational fields.

The Kepler problem arises in many contexts, some beyond the physics studied by Kepler himself. The Kepler problem is important in celestial mechanics, since Newtonian gravity obeys an inverse square law. Examples include a satellite moving about a planet, a planet about its sun, or two binary stars about each other. The Kepler problem is also important in the motion of two charged particles, since Coulomb's law of electrostatics also obeys an inverse square law. Examples include the hydrogen atom, positronium and muonium, which have all played important roles as model systems for testing physical theories and measuring constants of nature.

The **Kepler problem** and the **simple harmonic oscillator problem** are the two most fundamental problems in classical mechanics. They are the only two problems that have closed orbits for every possible set of initial conditions, i.e., return to their starting point with the same velocity (**Bertrand's theorem**). The Kepler problem is often used to develop new methods in classical mechanics, such as **Lagrangian mechanics**, **Hamiltonian mechanics**, the **Hamilton–Jacobi equation**, and **action-angle coordinates**. The Kepler problem also conserves the **Laplace–Runge–Lenz vector**, which has since been generalized to include other interactions. The solution of the Kepler problem allowed scientists to show planetary motion could be explained entirely by classical mechanics and Newton's law of gravity; the scientific explanation of planetary motion played an important role in ushering in the Enlightenment.

Mathematical definition: The central force $\vec{F}$ that varies in strength as the inverse square of the distance r between them: $\vec{F} = \frac{k}{r^2}\hat{r}$, where k is a constant and $\hat{r}$ represents the unit vector along the line between them. The force may be either attractive ($k < 0$) or repulsive ($k > 0$). The corresponding scalar potential (the potential energy of the non-central

body) is: $V(r) = \frac{k}{r}$.

Kepler Problem: $\begin{cases} \dot{q} &= p \\ \dot{p} &= k\frac{q}{|q|^3} \end{cases}$. Let $\Delta = \{q : q_i = q_j, \text{ where } i \neq j\}$. Solutions exist for the Kepler problem on $\mathbb{R}^2 \backslash \Delta$. That is, away from all collisions. However, you can regularize collisions of 2 bodies. Then, solutions exist for all time.

Bertrand's Theorem

In classical mechanics, Bertrand's theorem states that among central-force potentials with bounded orbits, there are only two types of central force potentials with the property that all bounded orbits are also closed orbits: (1) an inverse-square central force such as the gravitational or electrostatic potential: $V(r) = -\frac{k}{r}$, and (2) the radial harmonic oscillator potential: $V(r) = \frac{1}{2}kr^2$. The theorem was discovered by and named for the French mathematician Joseph Bertrand (1822–1900).

3-Body Problem (Existence and Uniqueness):

The mathematical statement of the three-body problem can be given in terms of the Newtonian equations of motion for vector positions $r_i = (x_i, y_i, z_i)$ of three gravitationally interacting bodies with masses m_i:

$$\ddot{r}_1 = -Gm_2\frac{r_1 - r_2}{|r_1 - r_2|^3} - Gm_3\frac{r_1 - r_3}{|r_1 - r_3|^3}, \qquad \ddot{r}_2 = -Gm_3\frac{r_2 - r_3}{|r_2 - r_3|^3} - Gm_1\frac{r_2 - r_1}{|r_2 - r_1|^3}, \qquad \ddot{r}_3 = -Gm_2\frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_2\frac{r_3 - r_2}{|r_3 - r_2|^3},$$

where G is the gravitational constant. This is a set of 9 second-order differential equations. The problem can also be stated equivalently in the Hamiltonian formalism, in which case it is described by a set of 18 first-order differential equations, one for each component of the positions r_i and momenta p_i: $\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial r_i}$, where H is the Hamiltonian:

$$H = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3}\right) - G\left(\frac{m_1 m_2}{|r_1 - r_2|} + \frac{m_2 m_3}{|r_3 - r_2|} + \frac{m_3 m_1}{|r_3 - r_1|}\right).$$

In this case H is simply the total energy of the system, gravitational plus kinetic.

Existence and Uniqueness of Solutions: We run into problems when there are collisions of the gravitationally attractive bodies causing singularities in the above differential equations.

Regularization Theory: Aim is to transform singular differential equations into regular ones, thus providing an efficient mathematical tool to analyze motions leading to collisions.

It has been shown that one can regularize double collisions, but not triple collisions. So, the solutions exist as long as we avoid triple collisions. Be wary of non-collision singularities!

Polar Coordinates:

Let $\vec{r} = (x, y) = r(\cos\varphi, \sin\varphi)$, so that $\vec{v} = \dot{r}\,(\cos\varphi, \sin\varphi) + r\,\dot{\varphi}\,(-\sin\varphi, \cos\varphi)$, and

$$\vec{a} = \ddot{r}\,(\cos\varphi, \sin\varphi) + 2\,\dot{r}\dot{\varphi}\,(-\sin\varphi, \cos\varphi) + r\,\ddot{\varphi}\,(-\sin\varphi, \cos\varphi) - r\,\dot{\varphi}^2\,(\cos\varphi, \sin\varphi). \qquad [0]$$

If we let $\hat{r} := (\cos\varphi, \sin\varphi)$, and $\hat{\varphi} := \frac{d\hat{r}}{d\varphi} = (-\sin\varphi, \cos\varphi)$ represent a new basis for our space (note $\hat{r} \cdot \hat{\varphi} = 0$), then we have $\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\varphi}\,\hat{\varphi}$, and $\vec{a} = \left(\ddot{r} - r\,\dot{\varphi}^2\right)\hat{r} + \left(2\,\dot{r}\dot{\varphi} + r\,\ddot{\varphi}\right)\hat{\varphi}$.

Calculating Equations of Motion - Newton

From Newton's **2nd Law of Motion** and **Law of Universal Gravitation**, we have $\vec{F} = m\vec{a} = -G\frac{Mm}{r^2}\hat{r}$ or $\vec{a} = -\frac{GM}{r^2}\hat{r}$.

Substituting in [0], we have: $\left(\ddot{r} - r\,\dot{\varphi}^2\right)\hat{r} + \left(2\,\dot{r}\dot{\varphi} + r\,\ddot{\varphi}\right)\hat{\varphi} = -\frac{GM}{r^2}\hat{r}$.

Separately equating the radial and angular components, gives us **equations of motion** (radial and angular, resp.):

$$\ddot{r} - r\,\dot{\varphi}^2 = -\frac{GM}{r^2} \text{ and } 2\,\dot{r}\dot{\varphi} + r\,\ddot{\varphi} = 0 \qquad [1]$$

Note that the equations are coupled 2nd ODEs. To understand what the equations mean, first we take a look at angular momentum: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$. Using our velocity equation from above, we have:
$\vec{L} = \vec{r} \times m\left(\dot{r}\,\hat{r} + r\,\dot{\varphi}\,\hat{\varphi}\right) = m\left(r\hat{r} \times \left(\dot{r}\,\hat{r} + r\,\dot{\varphi}\,\hat{\varphi}\right)\right)$, and noting that $\hat{r} \times \hat{\varphi} = \hat{k}$ and $\hat{r} \times \hat{r} = 0$, gives us:
$\vec{L} = m\left(r^2\,\dot{\varphi}\right)\hat{k}$. And taking the derivative, we have: $\frac{d}{dt}\vec{L} = m\left(2r\,\dot{r}\dot{\varphi} + r^2\,\ddot{\varphi}\right)\hat{k}$. And since for central forces, we have that the angular momentum is constant, we see that $2\,\dot{r}\dot{\varphi} + r\,\ddot{\varphi} = 0$ (for nonzero radii) which is the 2nd of the equations of motion in [1]. So, [1] can be replaced with the requirement that the angular momentum is constant, and $L = mr^2\,\dot{\varphi}$. Solving for the rotation speed $\dot{\varphi}$, we have: $\dot{\varphi} = \frac{L}{mr^2}$, [2]

Substituting back into the first equation of [1], we have: $\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2}$. [3]

Thus, conservation of angular momentum "de-couples" the equations of motion. The equation has only one variable *r*, so it is often called a "one-dimensional equation." But you should always keep in mind that the motion takes place in two dimensions. Now we have the system in a form we can solve for, since [3] is an equation only in *r*. Once we solve for *r* (using initial conditions) we can plug it into [2], and solve for φ, using initial conditions.

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Clear[plot]; plot[t_] = NDSolve[{r"[t] - ((7^2)/(2^2*r[t])) == -((70)/(r[t]^2)), r[0] == 1, r'[0] == 8.31}, r[t], {t, 0, 1}][[1, 1, 2]];
Plot[plot[t], {t, 0, 4}]
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Lagrangian Method

For an alternative method of calculating the **equations of motion**, we use the Lagrangian: $\mathcal{L} = T - U$, where *U* is the potential energy and *T* is the kenetic.

Recall that force can be defined as the negative of the vector gradient of the potential field: $\vec{F}\left(\vec{r}\right) = -\frac{dU}{d\vec{r}}$.

For a particle of mass *m* gravitationally attracted to a body of mass *M* :
$U\left(\vec{r}\right) = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_{\infty}^{\vec{r}} -\frac{GMm}{r^2}\hat{r} \cdot d\vec{r} = -\frac{GMm}{r}$.

The kinetic energy for this two-dimensional problem is $T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, or in polar coordinates:
$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right)$. And the Lagrangian is: $\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + \frac{GMm}{r}$.

Recall that the Lagrange **equations of motion** have the form: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$, where the q_i'*s* in our case are *r* and θ.
So we have the two equations: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0$ and $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$. Taking the appropriate derivatives, this gives us:
$\frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{GMm}{r}\right) - \frac{\partial}{\partial r}\left(\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{GMm}{r}\right)$

$= \frac{d}{dt}\left(m\,\dot{r}\right) - mr\,\dot{\theta}^2 + \frac{GMm}{r^2} = 0$, and

$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{GMm}{r}\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{GMm}{r}\right) = \frac{d}{dt}\left(mr^2\,\dot{\theta}\right) = \frac{dL}{dt} = 0.$

This last equation gives us $\dot{\theta} = \frac{L}{mr^2}$. Substituting it into the first equation, we find: $\ddot{r} - \frac{L^2}{m^2r^3} + \frac{GM}{r^2} = 0$. Same as above, but much more easily computed than using Newton's 2nd law.

Hamiltonian of Newtonian N-Body Problem:

The n-body problem considers n point masses m_i, $i = 1, 2, \ldots, n$ in an inertial reference frame in three dimensional space $\mathbb{R}^3$ moving under the influence of mutual gravitational attraction. Each mass m_i has a position vector q_i.

Newton: His second law says mass times acceleration $m_i \frac{d^2 q_i}{dt^2}$ is equal to the sum of forces on the mass. Newton's law of gravity says gravitational force felt on mass m_i by a single mass m_j is given by: $F_{ij} = \frac{G m_i m_j (q_j - q_i)}{|q_j - q_i|^3}$ (metric induced by the l_2 norm). Summing over all masses yields the n-body equations of motion: $m_i \frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j (q_j - q_i)}{|q_j - q_i|^3} = -\frac{\partial U}{\partial q_i}$, where U is the self-potential energy: $U := -\sum_{1 \leq i < j \leq n} \frac{G m_i m_j}{|q_j - q_i|}$. A system of $3n$ second order ODEs, with $6n$ initial conditions as 3n initial position coordinates and 3n initial momentum values.

Hamilton: However, defining the momentum to be $p_i := m_i \frac{dq_i}{dt}$, we observe the kinetic energy is $T = \sum_{i=1}^{n} \frac{1}{2} m_i v^2 = \sum_{i=1}^{n} \frac{|p_i|^2}{2m_i}$, the Hamiltonian function is $H = T + U$, and Hamilton's equations of motion for the n-body problem are: $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$, $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$. Hamilton's equations show that the n-body problem is a system of 6n first-order differential equations.

Symmetries in the n-body problem yield global **integrals of motion** that simplify the problem. **Translational symmetry** of the problem results in the center of mass: $C = \frac{\Sigma m_i q_i}{\Sigma m_i}$, moving with constant velocity, so that $C = L_0 t + C_0$, where L_0 is the linear velocity and C_0 is the initial position. The constants of motion L_0 and C_0 represent six integrals of motion. **Rotational symmetry** results in the total angular momentum being constant: $A = \Sigma(q_i \times p_i)$. The three components of the total angular momentum A yield three more constants of motion. The last general constant of motion is given by the **conservation of energy** H. Hence, every n-body problem has ten integrals of motion.

Because T and U are homogeneous functions of degree 2 and -1, respectively, the equations of motion have a **scaling invariance**: if $q_i(t)$ is a solution, then so is $\lambda^{-\frac{2}{3}} q_i(\lambda t)$ for any $\lambda > 0$.

The **moment of inertia** of an n-body system is given by: $I = \Sigma m_i q_i \cdot q_i = \Sigma m_i |q_i|^2$. Then the **Lagrange–Jacobi formula** states that: $\frac{d^2 I}{dt^2} = 2T - U = T + h$, where h is the total energy of the system.

For systems in dynamic equilibrium, the longterm time average of $\left\langle \frac{d^2 I}{dt^2} \right\rangle$ is zero. Then on average the total kinetic energy is half the total potential energy, $\langle T \rangle = \frac{1}{2} \langle U \rangle$, which is an example of the virial theorem for gravitational systems. If M is the total mass and R a characteristic size of the system (for example, the radius containing half the mass of the system), then the critical time for a system to settle down to a dynamic equilibrium is: $t_{cr} = \sqrt{\frac{GM}{R^3}}$.

Jacobi Coordinates (NBP):

In the theory of many-particle systems, Jacobi coordinates often are used to simplify the mathematical formulation. These coordinates are particularly common in treating polyatomic molecules and chemical reactions, and in celestial mechanics. An algorithm for generating the Jacobi coordinates for N bodies may be based upon binary trees. In words, the algorithm is described as follows:

> Let m_j and m_k be the masses of two bodies that are replaced by a new body of virtual mass $M = m_j + m_k$. The position coordinates x_j and x_k are replaced by their relative position $r_{jk} = x_j - x_k$ and by the vector to their center of mass $R_{jk} = (m_j q_j + m_k q_k)/(m_j + m_k)$. The node in the binary tree corresponding to the virtual body has m_j as its right child and m_k as its left child. The order of children indicates the relative coordinate points from x_k to x_j. Repeat the above step for $N - 1$ bodies, that is, the $N - 2$ original bodies plus the new virtual body.

For the N-body problem the result is: $r_j = \frac{1}{m_{0j}} \sum_{k=1}^{j} m_k x_k - x_{j+1}$, $r_N = \frac{1}{m_{0N}} \sum_{k=1}^{N} m_k x_k$, with $m_{0j} = \sum_{k=1}^{j} m_k$.

The vector r_N is the center of mass of all the bodies. The result one is left with is thus a system of $N - 1$ translationally invariant coordinates $r_1, \ldots, r_{N-1}$ and a center of mass coordinate r_N, from iteratively reducing two-body systems within the many-body system.

Symmetries ↔ Conserved Quantities (NBP):

Given a collection of point-particles, interacting through an attractive force $\sim \frac{1}{r^2}$. Knowing only $m_1 a_i = \sum_i \frac{G m_1 m_i}{r_i^2}$ and initial conditions we can deduce the motion of the system. Consequently we can observe that three quantities remains constant: A) center of mass of the system, B) total energy, and C) angular momentum.

A) Center of mass is the most basic, it needs just Newton's laws:
Second Law: $m_i \ddot{\vec{r}}_i = F_i$, and Third Law: $\sum_i F_i = 0$.
Summing over *i* one obtains: $\frac{d^2}{dt^2}(\sum_i m_i r_i) = \sum_i F_i = 0$.
So we have $\frac{\sum_i m_i r_i}{\sum_i m_i} = c_1 t + c_2$, but by the symmetry of translation invariance, we can choose a moving inertial reference frame such that $\frac{\sum_i m_i r_i}{\sum_i m_i} = 0$.

B) For the total energy you need the force/potential to be independent of time:
$F_i = -\frac{d}{dr_i} U(r_1, r_2, \dots)$. Then just take the total energy : $E = \sum_i \frac{m_i |\dot{\vec{r}}_i|^2}{2} + U$.
And differentiate with respect to time: $\frac{dE}{dt} = \sum_i m_i (\ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i) + \sum_i \frac{dU}{dr_i} \dot{\vec{r}}_i = (F - F) \cdot \dot{\vec{r}}_i = 0$.

C) Finally, for the angular momentum, *U* must have the symmetry of rotational invariance:
$U(R\vec{r}_1, R\vec{r}_2, R\vec{r}_3, \dots) = U(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots)$, where *R* is a rotation matrix. Consider now very small (infinitesimal) rotation: $R\vec{r}_i = \vec{r}_i + [\delta\varphi \times \dot{r}_i]$ for angle $\delta\varphi$. Substituting and expanding, one can get: $U + \delta\varphi \sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = U$. Which works for every angle $\delta\varphi$, so the following must hold $\sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = 0$. Finally take the angular momentum: $L = \sum_i m_i [\vec{r}_i \times \dot{\vec{r}}_i]$, and differentiate with respect to time:
$\frac{dL}{dt} = \sum i [\vec{r}_i \times m_i \ddot{\vec{r}}_i] + \sum_i m_i [\dot{\vec{r}}_i \times \dot{\vec{r}}_i] = \sum_i [-m_i \ddot{\vec{r}}_i \times \vec{r}_i] + 0 = \sum_i [-F_i \times \vec{r}_i] = \sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = 0$.

Levi-Civita Transformation (**NBP**): http://assets.press.princeton.edu/chapters/s7687.pdf

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Central Configurations of the Newtonian n-Body Problem

Central Force: Suppose that a force acting on a particle of mass *m* has the properties:

- The forces always directed from *m* toward, or away, from a fixed point *O*,
- The magnitude of the force only depends on the distance *r* from *O*.

Forces having these properties are called central forces

Mathematically, $\vec{F}$ is a central force if and only if: $\vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}$, where $\hat{r} = \frac{\vec{r}}{r}$ is a unit vector in the direction of $\vec{r}$, (the position vector of the particle).

If $f(r) < 0$, the force is said to be attractive, and if $f(r) > 0$, the force is said to be repulsive.

Properties of a particle moving under the influence of a central force:

1. The path of the particle must be a **plane curve**
2. The angular momentum of the particle is conserved, i.e., it is constant in time.
3. The particle moves in such a way that the position vector sweeps out equal areas in equal times.

   In other words, the time rate of change in area is constant. This is referred to as the **Law of Areas**.

Angular Momentum is Constant / Central Force Motion is Planar

Let's show that the motion of a particle under a central force $\vec{F}$ always remains in the plane defined by its initial position and velocity. We show that $\vec{F}$ lies in this plane. And since the position $\vec{r}$, velocity $\vec{v}$ and force $\vec{F}$ all lie in the same plane, there is never an acceleration perpendicular to that plane. To this end, it suffices to show that the angular momentum $\vec{L}$

$= \vec{r} \times \vec{p}$ of the particle is constant, where $\vec{p}$ is its linear momentum. If $\vec{F}$ were not in the plane, then this would imply that the force would move $\vec{r}, \vec{v}$ to form a new plane, and $\vec{L}$, being perpendicular to the new plane, would change. Recall that the time derivative of the angular momentum is:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\vec{v} \times m\vec{v}) + \left(\vec{r} \times \frac{d}{dt}\vec{p}\right) = \vec{r} \times \vec{F} = \text{net torque.}$$

When $\vec{F}$ is a central force, the remaining term $\vec{r} \times \vec{F}$ is also zero because the vectors $\vec{r}$ and $\vec{F}$ point in the same or opposite directions. Therefore, the angular momentum $\vec{L}$ is constant:

$$\frac{d\vec{L}}{dt} = \vec{r} \times f(r)\hat{r} = r\hat{r} \times f(r)\hat{r} = rf(r)(\hat{r} \times \hat{r}) = 0.$$

Another way of looking at this is to recall that the time rate of change of angular momentum is equal to the torque. A central force cannot exert a torque on a particle. These simple physical arguments lead us to an important conservation law concerning the motion of any particle under the action of any central force: both the magnitude and the direction of the angular momentum are constant. Also note that: $\vec{r} \cdot \vec{L} = \vec{r} \cdot \left(\vec{r} \times \vec{p}\right) = \vec{p} \cdot \left(\vec{r} \times \vec{r}\right) = 0$. Consequently, the particle's position $\vec{r}$ (and hence velocity $\vec{v}$) always lies in a plane perpendicular to $\vec{L}$.

Central Force Fields are Conservative, Computing the Potential

To be considered conservative, the work done by a force $\vec{F}$ in moving a particle between two points must be independent of the path taken between the two points, i.e., it only depends on the endpoints of the path.

Potential energy is closely linked with forces. If the work done by a force on a body that moves from A to B does not depend on the path between these points (if the work is done by a conservative force), then the work of this force measured from A assigns a scalar value to every other point in space and defines a scalar potential field. In this case, the force can be **defined** as the negative of the vector gradient of the potential field. $\vec{F}\left(\vec{r}\right) = -\frac{dV}{d\vec{r}}$, which implies $V\left(\vec{r}_0\right) = -\int_{\infty}^{\vec{r}_0} \vec{F}\left(\vec{r}\right) \cdot d\vec{r}$ (Note that $\vec{F}(\infty) = 0$). So, the work done is: $W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = V\left(\vec{r}_1\right) - V\left(\vec{r}_2\right)$. Therefore, the work done depends only on the difference between the potential energy at the final position and the initial position, is independent of the path, and a central force is a conservative force. One consequence is that total energy is conserved.

But how do we compute the potential in a **central force field**? Note: $\vec{F}\left(\vec{r}\right) = -\frac{dV}{dr}$ implies $\vec{F} \cdot d\vec{r} = -dV$. [1]

Observe that since $\vec{r} \cdot \vec{r} = r^2$, and $d\left(\vec{r} \cdot \vec{r}\right) = d(r^2) \Rightarrow \left(\vec{r} \cdot d\vec{r}\right) + \left(d\vec{r} \cdot \vec{r}\right) = 2rdr$, we have: $\vec{r} \cdot d\vec{r} = rdr$. So evaluating the left-hand side of [1], we have: $\vec{F} \cdot d\vec{r} = f(r)\frac{\vec{r}}{r} \cdot d\vec{r} = f(r)dr$. Therefore, $f(r)dr = -dV$, from which it follows that: $V = -\int f(r)dr$. Hence, if we know the central force field $\vec{F} = f(r)\frac{\vec{r}}{r}$, this equation tells us how to compute the potential.

Specific Angular Momentum: The angular momentum divided by the mass, $\vec{h} := \frac{\vec{L}}{m}$, and

$$h := \frac{\left|\vec{L}\right|}{m} = r^2\,\dot{\varphi} = \left|\vec{r} \times \vec{v}\right|.$$

Constant Areal Velocity "Law of Areas"

Suppose that in time Δt, the position vector moves from $\vec{r}$ to $\vec{r} + \Delta\vec{r}$.

Claim: The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\vec{r}$ and $\Delta\vec{r}$.

Recall the Area of Parallelogram = height $\cdot \left|\vec{r}\right| = \left|\Delta\vec{r}\right| \sin\theta \left|\vec{r}\right| = \left|\vec{r} \times \Delta\vec{r}\right|$, where θ is the angle between $\vec{r}$ and $\Delta\vec{r}$. So, $\Delta A = \frac{1}{2}\left|\vec{r} \times \Delta\vec{r}\right|$.

One notes visually, that this shape roughly approximates the shape we are interested in, and as $\Delta t \to 0$, this approximation becomes exact. Dividing by Δt, and letting $\Delta t \to 0$, gives us:
$\dot{A} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} \left| \vec{r} \times \frac{\Delta r}{\Delta t} \right| = \frac{1}{2} \left| \vec{r} \times \vec{v} \right|$. To evaluate $\vec{r} \times \vec{v}$, we use $\vec{r} = r\hat{r}$, and we have:
$\vec{r} \times \vec{v} = \vec{r} \times \left(\dot{r}\, \hat{r} + r\, \dot{\theta}\, \hat{\theta} \right) = r\, \dot{r}\, (\hat{r} \times \hat{r}) + r^2\, \dot{\theta}\, \left(\hat{r} \times \hat{\theta} \right) = r^2\, \dot{\theta}\, \vec{k}$. Therefore, we have (via the specific angular momentum): $2\, \dot{A} = \left| \vec{r} \times \vec{v} \right| = r^2\, \dot{\theta} = h = \frac{|\vec{L}|}{m} =$ constant, and the vector: $\dot{\vec{A}} = \dot{A}\, \vec{k} = \frac{1}{2} r^2\, \dot{\theta}\, \vec{k}$, is called the **areal velocity**.

Total Energy

Since central forces are conservative, we know that total energy must be conserved. Now we derive expressions for the total energy of a particle of mass *m* in a central force field.

Direct Method: $E = K + V = \frac{1}{2} m v^2 + V = \frac{1}{2} m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) - \int f(r) dr =$ constant.

Equations of Motion Method: Here we realize the expression for the total energy as an integral of the equations of motion: $m \left(\ddot{r} - r\, \dot{\theta}^2 \right) = f(r)$ and $m \left(r\, \ddot{\theta} + 2\, \dot{r} \dot{\theta} \right) = 0$. We multiply the first equation by $\dot{r}$, the 2nd equation by $r\, \dot{\theta}$, and add them together to get:
$m \left(\dot{r} \ddot{r} + r^2\, \dot{\theta} \ddot{\theta} + r\, \dot{r} \dot{\theta}^2 \right) = f(r)\, \dot{r} = \dot{r}\, \frac{d}{dr} \int f(r) dr = \frac{dr}{dt} \frac{d}{dr} \int f(r) dr = \frac{d}{dt} \int f(r) dr$, or $\frac{1}{2} m \frac{d}{dt} \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) = \frac{d}{dt} \int f(r) dr$.
Integrating both sides of this equation with respect to time gives: $\frac{1}{2} m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) - \int f(r) dr = E =$ constant.

Regularization Theory

As the two bodies approach each other, their distance tends to zero and, consequently, the differential equation describing the dynamics of the system becomes singular when the two bodies collide. In the real world the point masses are replaced by bodies of finite (non zero) size; in this case, we refer to a binary collision whenever the distance between their centers of mass equals the sum of the radii. Even in the simplified approach that solar system objects are reduced to point masses, the description of the dynamics of an N–body system becomes difficult during close encounters, due to the loss of regularity of the equations of motion.

In order to get rid of this problem, regularization theories have been developed to transform a singular equation into a regular one. One such method is the so–called Levi–Civita regularization, which is based on an ad hoc transformation of the spatial variables and of time (through the introduction of a fictitious time), taking advantage of the conservation of energy. The description of the motion fails at the singularity, but what is even worse, it is rather difficult to investigate the dynamics in a neighbourhood of the singularity.

Even excluding a collision, it is anyway troublesome to explore the trajectories corresponding to close approaches: the integration step required in a numerical approach is usually very small, thus requiring a huge amount of computational time. The Levi–Civita regularization, which is based upon three main steps:

- the introduction of a suitable change of coordinates (typically the Levi–Civita transformation);
- a stretching of the time scale, to get rid of the fact that the velocity becomes infinite (namely, the introduction of a so–called fictitious time);
- the conservation of the energy, to transform the singular differential equations into regular ones (i.e., the study of the Hamiltonian system in the extended phase space).

Relative Equilibrium

A relative equilibrium in a symmetric dynamical system is a **group orbit** that is invariant under the dynamics. Another way of viewing a relative equilibrium is to consider the dynamics induced on the orbit space of the phase space, and then

a relative equilibrium is just an equilibrium point of this induced dynamics. For finite groups, relative equilibria are just (group orbits of) equilibria.

Relative equilibria are the symmetric analogues of fixed equilibria for dynamical systems which are invariant under the action of a Lie group. Examples include circular orbits in the Kepler problem, rigid bodies and heavy tops rotating about an axis of symmetry, and rotating systems of point vortices with constant geometric configurations.

A relative equilibrium of a Hamiltonian system with symmetry is a point of phase space giving an evolution which is a one-parameter orbit of the action of the symmetry group of the system. The evolutions of sufficiently small perturbations of a formally stable relative equilibrium are arbitrarily confined to that relative equilibrium's orbit under the isotropy subgroup of its momentum.

Given the correct initial velocities, a central configuration will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.

Relative equilibria and relative periodic solutions are related by symmetry reduction to equilibria and periodic solutions of the reduced dynamics.

A relative equilibrium is a solution which travels along an orbit of the symmetry group at constant speed.

Relative equilibria of the Nbody problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. A striking application of relative periodic orbits has been the discovery of "choreographies" in the N-body problems.

For the planar N-body problem, relative equilibria are uniformly rotating rigid solutions, and therefore, such configurations have a moment of inertia that is constant in time.

The modern story on equivariance and dynamical systems starts perhaps with S. Smale [10.26] and M. Field [10.27], and on bifurcations in presence of symmetries with Ruelle [10.28]. Ruelle proves that the stability matrix/Jacobian matrix evaluated at an equilibrium/fixed point $x \in \boldsymbol{\mathcal{M}}_G$ decomposes into linear irreducible representations of G, and that stable/unstable manifold continuations of its eigenvectors inherit their symmetry properties, and shows that an equilibrium can bifurcate to a rotationally invariant periodic orbit (i.e., relative equilibrium).

A relative equilibrium of a Hamiltonian system with symmetry is a point of phase space giving an evolution which is a one-parameter orbit of the action of the symmetry group of the system. The evolutions of sufficiently small perturbations of a formally stable relative equilibrium are arbitrarily confined to that relative equilibrium's orbit under the isotropy subgroup of its momentum. However, interesting evolution along that orbit, here called drift, does occur.

Stability of Relative Equilibria

Every planar CC determines a relative equilibrium solution of the N-body problem with every body moving on a circle with constant angular speed. In a uniformly rotating coordinate system these become equilibrium solutions and one can analyze their linear stability by finding the eigenvalues of the linearized differential equations. Strictly speaking, even the circular periodic orbits of the two-body problem are linearly unstable because they are part of a four-parameter family of RE solutions with different sizes, centers of mass and rotation speeds. This applies to all relative equilibria and is reflected in the existence of four eigenvalues with value 0 organized into two Jordan blocks. This trivial drifting apart of nearby relative equilibria can be eliminated by fixing the center of mass, energy and angular momentum. Then for linear stability it is necessary that the rest of the eigenvalues be purely imaginary numbers.

With this understanding, it is known (Gascheau (1842)) that Lagrange's equilateral triangle solutions are linearly stable provided the masses satisfy

$$27(m_1m_2 + m_3m_1 + m_2m_3) < (m_1 + m_2 + m_3)^2.$$

This holds only if there is a dominant mass, i.e., one mass must be much larger than the other two. It does hold in the case of the Sun, Jupiter and a third small body so, ignoring the other planets, a small body forming an equilateral triangle with the Sun and Jupiter could remain there for a long time. In fact there are clusters of small bodies found near these equilateral points: the Greek and Trojan asteroids. On the other hand, the collinear Eulerian relative equilibria are always linearly unstable. For example, the solution of Figure 7 is unstable and nearby solutions will diverge from it exponentially.

The RE based on the regular N-gon with N equal masses is linearly unstable as is the centered N-gon with $N + 1$ equal masses as in Figure 2. On the other hand a regular N-gon with a sufficiently large $(N + 1)$-st mass at the center is linearly stable, provided N is at least 7 (Moeckel (1995)). For small N, lumpy rings like the one in Figure 5 can be linearly stable.

James Clerk Maxwell studied the centered N-gon in connection with the stability of the rings of Saturn. He found that the mass ratio guaranteeing stability increases with N, so a given ring mass should not be split into too many pieces. This supported the idea that the rings were composed of discrete bodies instead of a continuously distributed dust or liquid. (Maxwell (1859)).

2BP wikipedia.org/wiki/Two-body_problem

Everything is a relative equilibrium.

Circular orbits of the 2 body problem are the simplest examples.

The two-body problem is to predict the motion of two massive objects which are abstractly viewed as point particles. The problem assumes that the two objects interact only with one another; the only force affecting each object arises from the other one, and all other objects are ignored.

The most prominent case of the classical two-body problem is the gravitational case (see also Kepler problem), arising in astronomy for predicting the orbits (or escapes from orbit) of objects such as satellites, planets, and stars. A two-point-particle model of such a system nearly always describes its behavior well enough to provide useful insights and predictions.

A simpler "one body" model, the "central-force problem", treats one object as the immobile source of a force acting on other. One then seeks to predict the motion of the single remaining mobile object. Such an approximation can give useful results when one object is much more massive than the other (as with a light planet orbiting a heavy star, where the star can be treated as essentially stationary).

However, the one-body approximation is usually unnecessary except as a stepping stone. For many forces, including gravitational ones, the general version of the two-body problem can be reduced to a pair of one-body problems, allowing it to be solved completely, and giving a solution simple enough to be used effectively.

By contrast, the three-body problem (and, more generally, the n-body problem for $n \geq 3$) cannot be solved in terms of first integrals, except in special cases.

Gravitation and other inverse-square examples

The two-body problem is interesting in astronomy because pairs of astronomical objects are often moving rapidly in arbitrary directions (so their motions become interesting), widely separated from one another (so they won't collide) and even more widely separated from other objects (so outside influences will be small enough to be ignored safely).

Under the force of gravity, each member of a pair of such objects will orbit their mutual center of mass in an elliptical pattern, unless they are moving fast enough to escape one another entirely, in which case their paths will diverge along other planar conic sections. If one object is very much heavier than the other, it will move far less than the other with reference to the shared center of mass. The mutual center of mass may even be inside the larger object.

For a mathematical summary of the solutions for this case, see Gravitational two-body problem. For the derivation of the solutions, see Classical central-force problem or Kepler problem.

In principle, the same solutions apply to macroscopic problems involving objects interacting not only through gravity, but through any other attractive scalar force field obeying an inverse-square law, with electrostatic attraction being the obvious physical example. In practice, such problems rarely arise. Except perhaps in experimental apparatus or other specialized equipment, we rarely encounter electrostatically interacting objects which are moving fast enough, and in such a direction, as to avoid colliding, and/or which are isolated enough from their surroundings.

Inapplicability to atoms and subatomic particles

Although the two-body model treats the objects as point particles, classical mechanics only apply to systems of macroscopic scale. Most behavior of subatomic particles cannot be predicted under the classical assumptions underlying

this article or using the mathematics here.

Electrons in an atom are sometimes described as "orbiting" its nucleus, following an early conjecture of Niels Bohr (this is the source of the term "orbital"). However, electrons don't actually orbit nuclei in any meaningful sense, and quantum mechanics are necessary for any useful understanding of the electron's real behavior. Solving the classical two-body problem for an electron orbiting an atomic nucleus is misleading and does not produce many useful insights.

Reduction to two independent, one-body problems

The complete two-body problem can be solved by re-formulating it as two one-body problems: a trivial one and one that involves solving for the motion of one particle in an external potential. Since many one-body problems can be solved exactly, the corresponding two-body problem can also be solved.

Let x_1 and x_2 be the vector positions of the two bodies, and m_1 and m_2 be their masses. The goal is to determine the trajectories $x_1(t)$ and $x_2(t)$ for all times t, given the initial positions $x_1(t = 0)$ and $x_2(t = 0)$ and the initial velocities $v_1(t = 0)$ and $v_2(t = 0)$.

When applied to the two masses, Newton's second law states that: $\vec{F}_{12}(x_1,x_2) = m_1 \ddot{x}_1$, [1]

and $\vec{F}_{21}(x_1,x_2) = m_2 \ddot{x}_2$, [2]

where F_{12} is the force on mass 1 due to its interactions with mass 2, and F_{21} is the force on mass 2 due to its interactions with mass 1. The two dots on top of the x position vectors denote their second derivative with respect to time, or their acceleration vectors.

Adding and subtracting these two equations decouples them into two one-body problems, which can be solved independently. Adding equations (1) and (2) results in an equation describing the center of mass (barycenter) motion. By contrast, subtracting equation (2) from equation (1) results in an equation that describes how the vector $r = x_1 - x_2$ between the masses changes with time. The solutions of these independent one-body problems can be combined to obtain the solutions for the trajectories $x_1(t)$ and $x_2(t)$.

Center of mass motion (1st one-body problem)

Let $\vec{R} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ be the position of the center of mass (**barycenter**) of the system. Addition of the force equations (1) and (2) yields: $m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = (m_1 + m_2) \ddot{R} = \vec{F}_{12} + \vec{F}_{21} = 0,$

where we have used Newton's third law $F_{12} = -F_{21}$ and where

$\ddot{R} = \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2}$.

The resulting equation: $\ddot{R} = 0$ shows that the velocity $V = dR/dt$ of the center of mass is constant, from which follows that the **total momentum** $m_1 v_1 + m_2 v_2$ is also constant (**conservation of momentum**). Hence, the position $R(t)$ of the center of mass can be determined at all times from the initial positions and velocities.

Displacement vector motion (2nd one-body problem)

Dividing both force equations by the respective masses, subtracting the second equation from the first and rearranging gives the equation: $\ddot{r} = \ddot{x}_1 - \ddot{x}_2 = \left(\frac{F_{12}}{m_1} - \frac{F_{21}}{m_2}\right) = (\frac{1}{m_1} + \frac{1}{m_2})F_{12}$ where we have again used Newton's third law $F_{12} = -F_{21}$ and where r is the displacement vector from mass 2 to mass 1, as defined above.

The force between the two objects, which originates in the two objects, should only be a function of their separation r and not of their absolute positions x_1 and x_2; otherwise, there would not be translational symmetry, and the laws of physics would have to change from place to place. The subtracted equation can therefore be written: $\mu \ddot{r} = F_{12}(x_1,x_2) = \vec{F}(\vec{r})$ where μ is the reduced mass $\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}$. Solving the equation for r(t) is the key to the two-body problem. The solution depends on the specific force between the bodies, which is defined by $\vec{F}(\vec{r})$. For the case where $\vec{F}(\vec{r})$ follows an inverse-square law, see the Kepler problem.

Once $R(t)$ and $r(t)$ have been determined, the original trajectories may be obtained: $\vec{x}_1(t) = \vec{R}(t) + \frac{m_2}{m_1 + m_2}\vec{r}(t)$, $\vec{x}_2(t) = \vec{R}(t) - \frac{m_1}{m_1 + m_2}\vec{r}(t)$ as may be verified by substituting the definitions of R and r into the right-hand sides of these two equations.

Gravitational two-body problem

The reduced mass multiplied by the relative acceleration between the two bodies is equal to the gravitational force ($\vec{F} = \frac{m_1 m_2}{m_1+m_2}\vec{a}_r$). The latter is proportional to the product of the two masses, which is equal to the reduced mass divided by the sum of the masses. Thus in the differential equation the two occurrences of the reduced mass cancel each other, and we get the same differential equation as for the position of a very small body orbiting a body with a mass equal to the sum of the two masses.

Assume: the vector $\vec{r}$ is the position of one body relative to the other

- $\vec{r}, \vec{v} \equiv \frac{d\vec{r}}{dt}$, the **semi-major axis** a, and the **specific relative angular momentu**m h are defined accordingly (hence r is the distance)
- h is the total **angular momentum** divided by the reduced mass
- $\mu = G(m_1 + m_2)$, the standard gravitational parameter (the sum of those for each mass) where: m_1 and m_2 are the masses of the two bodies.

Then:

- the general solution is (see also the orbit equation and two-body problem for an inverse-square force): $u(\theta) \equiv \frac{1}{r(\theta)} = \frac{\mu}{h^2}(1 + e\cos(\theta - \theta_0))$ for any nonnegative e, called the eccentricity; here θ is the true anomaly, which is the angle between the current position of the orbiting object and the location in the orbit at which it is closest to the central body (called the periapsis).
- the positions of the bodies with respect to the barycenter are $\frac{m_2}{m_1+m_2}$ and $\frac{m_1}{m_1+m_2}$ times r, respectively, so the two bodies' orbits are similar conic sections; the same ratios apply for the velocities, and, without the minus, for the angular momentum and for the kinetic energies, all with respect to the barycenter
- for circular orbits $rv^2 = r^3\omega^2 = \frac{4\pi^2 r^3}{T^2} = \mu$
- for elliptic orbits: $\frac{4\pi^2 a^3}{T^2} = \mu$ (with a expressed in AU and T in years, and with M the total mass relative to that of the Sun, we get $\frac{a^3}{T^2} = M$)
- for parabolic trajectories rv^2 is constant and equal to 2μ
- the formulas for the specific orbital energy ε apply, with specific potential and kinetic energy and their sum taken as the totals for the system, divided by the reduced mass; the kinetic energy of the smaller body is larger; the potential energy of the whole system is equal to the potential energy of one body with respect to the other, i.e. minus the energy needed to escape the other if the other is kept in a fixed position; this should not be confused with the smaller amount of energy one body needs to escape, if the other body moves away also, in the opposite direction: in that case the total energy the two need to escape each other is the same as the aforementioned amount; the conservation of energy for each mass means that an increase of kinetic energy is accompanied by a decrease of potential energy, which is for each mass the inner product of the force and the change in position relative to the barycenter, not relative to the other mass
- for elliptic and hyperbolic orbits $\mu = 2a|\varepsilon|$

For example, consider two bodies like the Sun orbiting each other:

- the reduced mass is one half of the mass of one Sun (one quarter of the total mass) at a distance of 1 AU: the **orbital period** is $\frac{\sqrt{2}}{2}$ year, the same as the orbital period of the Earth would be if the Sun had twice its actual mass; the total energy per kg reduced mass (90 MJ/kg) is twice that of the Earth–Sun system (45 MJ/kg); the total energy per kg total mass (22.5 MJ/kg) is one half of the total energy per kg Earth mass in the Earth–Sun system (45 MJ/kg)
- at a distance of 2 AU (each following an orbit like that of the Earth around the Sun): the orbital period is 2 years, the same as the orbital period of the Earth would be if the Sun would have one quarter of its actual mass at a distance of $\sqrt[3]{2} \approx 1.26$ AU: the orbital period is 1 year, the same as the orbital period of the Earth around the Sun. Similarly, a second Earth at a distance from the Earth equal to $\sqrt[3]{2}$ times the usual distance of geosynchronous orbits would be geosynchronous.

Central Configuration

A central configuration is a special arrangement of point masses interacting by Newton's law of gravitation with the following property: the gravitational acceleration vector produced on each mass by all the others should point toward the center of mass and be proportional to the distance to the center of mass. Central configurations (or CC's) play an important role in the study of the Newtonian N-body problem. For example, they lead to the only explicit solutions of the equations of motion, they govern the behavior of solutions near collisions, and they influence the topology of the integral manifolds.

The Basic CC Equations

Newton's law of motion for the gravitational N-body problem is: $m_i \ddot{q}_i = F_i = \sum_{j\neq i} \frac{m_i m_j (q_j - q_i)}{r_{ij}^3}$ (1)

Where $r_{ij} = |q_i - q_j|$. The force vector $F_i \in \mathbb{R}^d$ on the right-hand side can also be written as a partial gradient vector $F_i = \nabla_i U$ where: $U = \sum_{j\neq i} \frac{m_i m_j}{r_{ij}}$ (2)

is the Newtonian potential function and ∇_i denotes the vector of partial derivatives with respect to the d components of q_i .

The acceleration of the i-th body is $\frac{F_i}{m_i}$ so the condition for q to be a central configuration is:

$$\nabla_i U = -\lambda m_i (q_i - c) \quad (3)$$

where $c = \frac{m_1 q_1 + \ldots + m_N q_N}{m_1 + \ldots + m_N} \in \mathbb{R}^d$ (4)

is the center of mass and $\lambda \in R$ is a constant. By definition, $q \in \mathbb{R}_N^d$ is a central configuration for the masses m_i if and only if (3) and (4) hold for some constant λ.

It turns out, however, that **the values of λ and c are uniquely determined by (3)**.

> **Proof**: Note that by translation invariance and degree -1 homogeneity of the Newtonian potential we have:
> $\sum_i \nabla_i U = 0, \quad \sum_i q_i^T \nabla_i U = -U.$
> Together these give $\sum_i (q_i - c)^T \nabla_i U = -U$ and then (3) shows that $\lambda = U/I$, where: $I = \sum m_i |q_i - c|^2$ (5)
> is the moment of inertia with respect to c.
> Thus $\lambda > 0$ is uniquely determined. Finally, summing (3) shows that c must be the center of mass.

Thus q is a central configuration for the given masses if and only if (3) holds for some (unique): $\lambda \in \mathbb{R}, \ c \in \mathbb{R}^d$.

Equivalent Central Configurations and Normalized Equations

The central configuration equation (3) is invariant under the Euclidean similarities of $\mathbb{R}^d$ – translations, rotations, reflections and dilations. Call two configurations $q, q' \in \mathbb{R}^{Nd}$ equivalent if there are constants $k \in \mathbb{R}, b \in \mathbb{R}^d$ and a dxd orthogonal matrix Q such that $q'_i = kQq_i + b, \ i = 1, \ldots, N$. If q satisfies (3) with constants λ, c then q' satisfies (3) with constants $\lambda' = k^3\lambda, \ c' = c + b$. So one can speak of an equivalence class of central configurations.

Translation invariance can be used to eliminate the center of mass. For configurations with $c = 0$ the central configuration equations become:

$$-\lambda q_i = \sum_{j\neq i} \frac{m_j (q_j - q_i)}{r_{ij}^3} \quad (6)$$

and any configuration satisfying this equation has $c = 0$. Alternatively, substituting (4) into (3) leads, after some simplification, to

$$\sum_{j\neq i} m_j S_{ij} (q_j - q_i) = 0, \quad S_{ij} = \frac{1}{r_{ij}^3} - \lambda' \quad (7)$$

where $\lambda' = \frac{\lambda}{m_1 + \ldots + m_N}$.

Dilation invariance can be used to normalize the size of a central configuration. The moment of inertia (5) is a natural measure of the size and setting $I = 1$ is a popular normalization. Alternatively, one can normalize the size by fixing the value of λ in (6) or λ' in (7).

The Two-Body Problem

Any two configurations of $N = 2$ particles in $\mathbb{R}^d$ are equivalent. Moreover, (7) reduces to just one equation:

$S_{12}(q_1 - q_2) = 0 \qquad S_{12} = \frac{1}{r_{12}^3} - \lambda'$

which holds for $\lambda' = r_{12}^{-3}$. Thus every configuration of two bodies is central.

In this case, the possible motions are well-known – each mass moves on a conic section according to Kepler's laws. In particular, one has a family of elliptical periodic motions ranging from circular (eccentricity $\epsilon = 0$) to collinear (limit as $\epsilon \to 1$). The latter is an example of a total collision solution, that is, one for which all N bodies collide at the center of mass.

Symmetrical Configurations of Equal Masses

When all of the masses are equal, it is obvious from symmetry that certain configurations are central. In the plane ($d = 2$), one can place the masses at the vertices of a regular N-gon or at the vertices of a regular $(N - 1)$-gon with the last mass at the center (see Figure 2). Similarly, in space ($d = 3$), a regular polyhedron or centered regular polyhedron are central configurations but these are possible only for special values of N. For $d > 3$, regular polytopes are central configurations for certain values of N. Note however, that equal masses may admit other central configurations with less symmetry as in Figure 1 or with no symmetry at all as in Figure 3 .

When some but not all of the masses are equal it is again possible to look for central configurations which are symmetric under similarities of $\mathbb{R}^d$ which permute equal masses. For example, it is clear that the centered regular polygon is still a central configuration when the mass at the center is different from the others.

Euler

The first nontrivial examples of central configurations were discovered by Euler in 1767, who studied the case N=3, d=1, that is, three bodies on a line (Euler (1767)). When two masses are equal, one can get a central configuration by putting an arbitrary mass at their midpoint (a centered 2-gon). For three unequal masses it is not obvious that any central configurations exist. But Euler showed that, in fact, there will be exactly one equivalence class of collinear central configurations for each possible ordering of the masses along the line.

Lagrange

Lagrange found next example in the planar three-body problem $N = 3, d = 2$. Remarkably, an equilateral triangle is a central configuration, not only for equal masses, but for any three masses m_1, m_2, m_3. Moreover, it is the only noncollinear central configuration for the three-body problem (Lagrange (1772)).

When the masses are not equal, the center of mass will not be the center of the triangle and it is not at all obvious that the configuration is central. But it is easy to see it using mutual distance coordinates. The three mutual distances r_{12}, r_{31}, r_{23} can be used as local coordinates on the space of noncollinear configurations of three bodies in the plane up to symmetry. The potential and the moment of inertia can be expressed in these coordinates as

$U = \frac{m_1 m_2}{r_{12}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_3}{r_{23}} \qquad I = \frac{m_1 m_2 r_{12}^2 + m_3 m_1 r_{31}^2 + m_2 m_3 r_{23}^2}{m_1 + m_2 + m_3}.$

Now use the characterization of CC's as critical points of U with fixed I. Setting

$\frac{\partial U}{r_{ij}} + \frac{\lambda}{2}\frac{\partial I}{r_{ij}}$ gives $r_{ij}^3 = \frac{m_1 + m_2 + m_3}{\lambda}$, which holds for some λ if and only if the three distances are equal.

This result can be generalized to higher dimensions: the regular simplex is a CC of N bodies in $N - 1$ dimensions for all choices of the masses and is the only CC spanning the full $N - 1$ dimensions.

Properties of Central Configurations

- Released from rest, a c.c. maintains the same shape as it heads toward total collision (homothetic motion).
- Given the correct initial velocities, a c.c. will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit (elliptic, hyperbolic, parabolic, ejection-collision) can be attached to a c.c. to obtain a solution to the full n-body problem.
- For any collision orbit in the n-body problem, the colliding bodies asymptotically approach a c.c.

- Bifurcations in the topology of the integral manifolds (holding hc2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to central configurations.

Examples:

- Equilateral Triangle (Lagrange 1772)
- 3-Body Collinear Configuration (Euler 1767)
- Regular n-gon (equal mass required for $n \geq 4$)
- 1 + n-gon (arbitrary central mass) (Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings)

Lagrangian point wikipedia.org/wiki/Lagrangian_point

The points near two large bodies in orbit where a smaller object will maintain its position relative to the large orbiting bodies. At other locations, a small object would go into its own orbit around one of the large bodies, but at the Lagrangian points the gravitational forces of the two large bodies, the centripetal force of orbital motion, and (for certain points) the Coriolis acceleration all match up in a way that cause the small object to maintain a stable or nearly stable position relative to the large bodies.

There are five such points, labeled L1 to L5, all in the orbital plane of the two large bodies, for each given combination of two orbital bodies. For instance, there are five Lagrangian points L1 to L5 for the Sun–Earth system, and in a similar way there are five different Lagrangian points for the Earth–Moon system. L1, L2, and L3 are on the line through the centers of the two large bodies. L4 and L5 each form an equilateral triangle with the centers of the large bodies. L4 and L5 are stable, which implies that objects can orbit around them in a rotating coordinate system tied to the two large bodies.

Several planets have trojan satellites near their L4 and L5 points with respect to the Sun. Jupiter has more than a million of these trojans. Artificial satellites have been placed at L1 and L2 with respect to the Sun and Earth, and with respect to the Earth and the Moon. The Lagrangian points have been proposed for uses in space exploration.

History

The three collinear Lagrange points (L1, L2, L3) were discovered by Leonhard Euler a few years before Joseph-Louis Lagrange discovered the remaining two.

In 1772, Lagrange published an "Essay on the three-body problem". In the first chapter he considered the general three-body problem. From that, in the second chapter, he demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits.

L1 point

The L1 point lies on the line defined by the two large masses M1 and M2, and between them. It is the most intuitively understood of the Lagrangian points: the one where the gravitational attraction of M2 partially cancels M1's gravitational attraction.

Explanation

An object that orbits the Sun more closely than Earth would normally have a shorter orbital period than Earth, but that ignores the effect of Earth's own gravitational pull. If the object is directly between Earth and the Sun, then Earth's gravity counteracts some of the Sun's pull on the object, and therefore increases the orbital period of the object. The closer to Earth the object is, the greater this effect is. At the L1 point, the orbital period of the object becomes exactly equal to Earth's orbital period. L1 is about 1.5 million kilometers from Earth, or 0.01 au, 1/100th the distance to the Sun.

L2 point

The L2 point lies on the line through the two large masses, beyond the smaller of the two. Here, the gravitational forces of the two large masses balance the centrifugal effect on a body at L2.

Explanation

On the opposite side of Earth from the Sun, the orbital period of an object would normally be greater than that of Earth.

The extra pull of Earth's gravity decreases the orbital period of the object, and at the L2 point that orbital period becomes equal to Earth's. Like L1, L2 is about 1.5 million kilometers or 0.01 au from Earth.

L3 point

The L3 point lies on the line defined by the two large masses, beyond the larger of the two.

Explanation

Within the Sun–Earth system, the L3 point exists on the opposite side of the Sun, a little outside Earth's orbit and slightly further from the Sun than Earth is. This placement occurs because the Sun is also affected by Earth's gravity and so orbits around the two bodies' barycenter, which is well inside the body of the Sun. At the L3 point, the combined pull of Earth and Sun cause the object to orbit with the same period as Earth.

L4 and L5 points

The L4 and L5 points lie at the third corners of the two equilateral triangles in the plane of orbit whose common base is the line between the centers of the two masses, such that the point lies behind (L5) or ahead (L4) of the smaller mass with regard to its orbit around the larger mass.

The triangular points (L4 and L5) are stable equilibria, provided that the ratio of $\frac{M1}{M2}$ is greater than 24.96. This is the case for the Sun–Earth system, the Sun–Jupiter system, and, by a smaller margin, the Earth–Moon system. When a body at these points is perturbed, it moves away from the point, but the factor opposite of that which is increased or decreased by the perturbation (either gravity or angular momentum-induced speed) will also increase or decrease, bending the object's path into a stable, kidney bean-shaped orbit around the point (as seen in the corotating frame of reference).

In contrast to L4 and L5, where stable equilibrium exists, the points L1, L2, and L3 are positions of unstable equilibrium. Any object orbiting at L1, L2, or L3 will tend to fall out of orbit; it is therefore rare to find natural objects there, and spacecraft inhabiting these areas must employ station keeping in order to maintain their position.

Mathematical details

Lagrangian points are the constant-pattern solutions of the restricted three-body problem. For example, given two massive bodies in orbits around their common barycenter, there are five positions in space where a third body, of comparatively negligible mass, could be placed so as to maintain its position relative to the two massive bodies. As seen in a rotating reference frame that matches the angular velocity of the two co-orbiting bodies, the gravitational fields of two massive bodies combined providing the centripetal force at the Lagrangian points, allowing the smaller third body to be relatively stationary with respect to the first two.

L1

The location of L1 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} = \frac{M_2}{r^2} + \frac{M_1}{R^2} - \frac{r(M_1+M_2)}{R^3}$, where r is the distance of the L1 point from the smaller object, R is the distance between the two main objects, and M1 and M2 are the masses of the large and small object, respectively. Solving this for r involves solving a quintic function, but if the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then L1 and L2 are at approximately equal distances r from the smaller object, equal to the radius of the Hill sphere, given by: $r \approx R\sqrt[3]{\frac{M_2}{3M_1}}$. This distance can be described as being such that the orbital period, corresponding to a circular orbit with this distance as radius around M2 in the absence of M1, is that of M2 around M1, divided by $\sqrt{3} \approx 1.73$: $T_{s,M_2}(r) = \frac{T_{M_2,M_1}(R)}{\sqrt{3}}$.

L2

The location of L2 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} + \frac{M_2}{r^2} = \frac{M_1}{R^2} + \frac{r(M_1+M_2)}{R^3}$, with parameters defined as for the L1 case. Again, if the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then L2 is at approximately the radius of the Hill sphere, given by: $r \approx R\sqrt[3]{\frac{M_2}{3M_1}}$.

L3

The location of L3 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} + \frac{M_2}{(2R-r)^2} = \frac{M_1}{R^2} + \frac{M_1+M_2}{R^2} - \frac{r(M_1+M_2)}{R^3}$ where r now indicates the distance of L3 from the "antipodal position" of the smaller object. If the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then: $r \approx R + R\frac{5M_2}{12M_1}$.

L4 and L5

The reason these points are in balance is that, at L4 and L5, the distances to the two masses are equal. Accordingly, the gravitational forces from the two massive bodies are in the same ratio as the masses of the two bodies, and so the resultant force acts through the barycenter of the system; additionally, the geometry of the triangle ensures that the resultant acceleration is to the distance from the barycenter in the same ratio as for the two massive bodies. The barycenter being both the center of mass and center of rotation of the three-body system, this resultant force is exactly that required to keep the smaller body at the Lagrange point in orbital equilibrium with the other two larger bodies of system. (Indeed, the third body need not have negligible mass.) The general triangular configuration was discovered by Lagrange in work on the three-body problem.

Stability

Although the L1, L2, and L3 points are nominally unstable, there are (unstable) periodic orbits called "halo" orbits around these points in a three-body system. A full n-body dynamical system such as the Solar System does not contain these periodic orbits, but does contain quasi-periodic (i.e. bounded but not precisely repeating) orbits following Lissajous-curve trajectories. These quasi-periodic Lissajous orbits are what most of Lagrangian-point space missions have used until now. Although they are not perfectly stable, a modest effort of station keeping keeps a spacecraft in a desired Lissajous orbit for a long time. Also, for Sun–Earth-L1 missions, it is preferable for the spacecraft to be in a large-amplitude (100,000–200,000 km or 62,000–124,000 mi) Lissajous orbit around L1 than to stay at L1, because the line between Sun and Earth has increased solar interference on Earth–spacecraft communications. Similarly, a large-amplitude Lissajous orbit around L2 keeps a probe out of Earth's shadow and therefore ensures continuous illumination of its solar panels.

The L4 and L5 points are stable provided that the mass of the primary body (e.g. the Earth) is at least 25 times the mass of the secondary body (e.g. the Moon). The Earth is over 81 times the mass of the Moon (the Moon is 1.23% of the mass of the Earth). Although the L4 and L5 points are found at the top of a "hill", as in the effective potential contour plot above, they are nonetheless stable. The reason for the stability is a second-order effect: as a body moves away from the exact Lagrange position, Coriolis acceleration (which depends on the velocity of an orbiting object and cannot be modeled as a contour map) curves the trajectory into a path around (rather than away from) the point.

Denjoy Theorem

Denjoy theorem gives a sufficient condition for a diffeomorphism f of the circle to be topologically conjugate ($\widetilde{f} = g^{-1}fg$) to a diffeomorphism $\widetilde{f}$ of a special kind, namely an irrational rotation. Denjoy proved the theorem in the course of his topological classification of homeomorphisms of the circle. He also gave an example of a C^1 diffeomorphism with an irrational rotation number that is not conjugate to an irrational rotation.

Statement of the theorem

Let $f : S^1 \to S^1$ be an orientation-preserving diffeomorphism of the circle whose rotation number $\theta = \rho(f)$ is irrational. Assume that it has positive derivative $f'(x) > 0$ that it is a continuous function with bounded variation on the interval $[0, 1)$. Then f is topologically conjugate to the irrational rotation by θ. Moreover, every orbit is dense and every nontrivial interval I of the circle intersects its forward image $f^q(I)$, for some $q > 0$ (this means that the non-wandering set of f is the whole circle).

Complements

If f is a C^2 map, then the hypothesis on the derivative holds; however, for any irrational rotation number Denjoy constructed an example showing that this condition cannot be relaxed to C^1.

If the rotation number is irrational, it can be shown that (under the additional assumption that both the map f and its inverse are C^2) the dynamics are topological equivalent to a rigid rotation through angle ρ; thus, the dynamics are non-chaotic and the forward orbit of any initial condition eventually fills the whole circle. In contrast, if $\rho = \frac{p}{q}$ is rational, then the dynamics are said to be mode locked and there is at least one orbit of period q. Typically, there will be two such orbits, with one stable and one unstable. Given a family of circle maps parameterized by α, then the rotation number will generically be rational over intervals of α-values (e.g. $\rho = \frac{p}{q}$ while $\alpha \in (a, b)$). Both irrational and rational rotations occur for sets of α-values that have positive measure ($\#\{\alpha : \rho(f_\alpha) \in \mathbb{R}\backslash\mathbb{Q}\} > 0$ and $\#\{\alpha : \rho(f_\alpha) \in \mathbb{Q}\} > 0$).

From the previous result (?), we know that such a map cannot have a periodic point. One such map is the rotation map $\tau_\omega(\theta) = \theta + 2\pi\omega$, where ω is irrational. One might be hard-pressed to think of another example of a homeomorphism with irrational rotation number, but there are many examples of such maps which are not topologically conjugate to an irrational rotation. Recall that, by Jacobi's Theorem, all orbits of an irrational rotation are dense in S^1. This property must be shared by any map topologically conjugate to an irrational rotation. So, to produce a different map of S^1, we need only find a map which has an orbit that is not dense. The following example, due to Denjoy, shows how to manufacture such a map.

Example (**A Denjoy map**): We will perform "surgery" on τ_ω where ω is irrational. Take any point $\theta_0 \in S^1$. We cut out each point on the orbit of θ_0 and replace it with a small "interval." That is, at the point $\tau_\omega^n(\theta_0)$ we cut apart the circle and glue in a small interval I_n in its place. Provided we take I_n small enough so that the length $\ell(I_n)$ are positive and satisfy $\sum_{n=-\infty}^{\infty} \ell(I_n) < \infty$, the result of this "operation" is still a"Circle" - a little larger than before but still a simple closed curve. We now extend the map to the union of the I_n's by choosing any orientation preserving diffeomorphism h_n taking I_n to I_{n+1}. This extends our original map to be a homeomorphism of the new circle. Note that the new map has no periodic points, so its rotation number is irrational. Moreover, no point in the interior of I_n ever returns to I_n under iteration of the map, so the orbits of these points are certainly not dense.

Remark

The Donjoy example is clearly a homeomorphism of the circle, but its construction probably casts doubt in the readers mind as to whether it can be made into a diffeomorphism. Actually, one may choose the h_n's carefully enough so that the resulting map is a C^1diffeomorphism. This construction however, cannot yield a C^2 diffeomorphism. It is known that a C^2 diffeomorphism with irrational rotation number is always topologically conjugate to τ_ω for appropriate ω. Thus, there are surprisingly complicated differences between the dynamics of C^1 and C^2 diffeomorphisms of the circle.

Regarding the family of maps parameterized by μ, is it related to this theorem:

Suppose $f : S^1 \to S^1$ is an orientation preserving diffeomorphism. Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $g : S^1 \to S^1$ is also a diffeomorphism which is C^0-δ close to f, then $|\rho(f) - \rho(g)| < \varepsilon$.

Dynamical Systems Notes: Math 8501/2

Taught by Arnd Scheel

Chapter 1: Examples of Differential Equations, Basic Concepts

Examples: $y' = 1 - 3t + y + t^2 + ty$ with $y(0) = 0$.

To solve, we are going to use the ansatz (power series): $y_0 = 0 + y_1 t + y_2 t^2 + \dots$ (constant term 0 since $y(0) = 0$). Taking the derivative and substituting in, we have:

$$y_0' = y_1 + 2y_2 t + \dots \quad \Rightarrow \quad y_1 + 2y_2 t + \dots = 1 - 3t + (y_1 t + y_2 t^2 + \dots) + t^2 + t(y_1 t + y_2 t^2 + \dots)$$
$$= 1 + (y_1 - 3)t + (y_2 + 1 + y_1)t^2 + O(t^3), \text{ so that } y_1 = 1.$$

So, our ansatz becomes $y_0 = t + y_2 t^2 + \dots$ Substituting in again, we have:

$$y_0' = 1 + 2y_2 t + \dots \quad \Rightarrow \quad 1 + 2y_2 t + \dots = 1 - 3t + (t + y_2 t^2 + \dots) + t^2 + t(t + y_2 t^2 + \dots)$$
$$= 1 - 2t + (y_2 + 2)t^2 + O(t^3), \text{ so that } y_2 = -1, \text{ and so forth.}$$

Oscillators

Harmonic: In classical mechanics, a harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force $\vec{F}$ proportional to the displacement $\vec{x}$: For example the linear force $\vec{F} = -k\vec{x}$, where k is a positive constant. If $\vec{F}$ is the only force acting on the system, the system is called a **simple** (**linear**) **harmonic oscillator**, and it undergoes simple harmonic motion: sinusoidal oscillations about the equilibrium point, with a constant amplitude and constant frequency (which does not depend on amplitude).

Nonlinear Oscillators - **Pendulum of length** ℓ: $\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi$ in one dimension, or more generally $M\ddot{x} = -\nabla V(x)$ where $x \in \mathbb{R}^n$.

Planetary Motion: N-body problem with point-masses $x = (x^1, x^2, \dots, x^N)$ where $x^j \in \mathbb{R}^3$, and the potential is $V(x) = \sum_{1 \le j \ne k \le N} \frac{m_j m_k}{|x^j - x^k|}$. (negative potential pulling in the direction of the origin)

Friction: $\ddot{x} = -\nabla V(x) - \gamma \dot{x}$, for $\gamma > 0$.

Van der Pol Oscillator: In dynamics, the Van der Pol oscillator is a non-conservative oscillator with non-linear damping. It evolves in time according to the second-order differential equation: $\ddot{x} = \gamma(1 - x^2)\dot{x} - x = 0$, where x is the position coordinate—which is a function of the time t, and γ is a scalar parameter indicating the nonlinearity and the strength of the damping.

When x is small, the quadratic term x^2 is negligible and the system becomes a linear differential equation with a negative damping $-\gamma \dot{x}$. Thus, the fixed point $(x = 0, \dot{x} = 0)$ is unstable (an unstable focus when $0 < \gamma < 2$ and an unstable node, otherwise). On the other hand, when x is large, the term x^2 becomes dominant and the damping becomes positive. Therefore, the dynamics of the system is expected to be restricted in some area around the fixed point. Actually, the Van der Pol system satisfies the Liénard's theorem ensuring that there is a stable limit cycle in the phase space.

Predator Prey Model: The coupled equations: $x' = (A - By)x$, and $y' = (Cx - D)y$, where y is the predator, and x is the prey.

Mass Kinetics: $2H_2 + O_2 \to 2H_2O$. We model this using concentrations: $x_1 = [H_2]$, $x_2 = [O_2]$, $x_3 = [H_2O]$, and $x_1' = -2kx_1^2 x_2$, $x_2' = -kx_1^2 x_2$, $x_3' = 2kx_1^2 x_2$. Conservation laws simplify our systems, such as the conservation of mass in this case: $x_1 + x_3$ is constant. A variation of this is used to model political opinions in a population.

General ODE: $x' = f(t,x).$ $(*)$

Assume $x \in \mathcal{M}$, a manifold of dimension n, or even a Banach space $x \in X$. The right-hand side is a vector field on the manifold, that is $f(t,x) \in T_x\mathcal{M}$. An example is the pendulum equation, where $\varphi \in \mathcal{M}_0 = \mathbb{R}/2\pi\mathbb{Z}$, the circle with $T_\varphi\mathcal{M} = \mathbb{R}$. Higher order equations can be reduced to the form (*) by adding derivatives as additional variables, for instance setting $v = \dot{\varphi}$, we find: $\varphi' = v, \quad v' = -\frac{g}{\ell}\sin\varphi,$ where now $y = (\varphi, v) \in \mathcal{M} := \mathcal{M}_0 \times \mathbb{R}$.

In cases where $f(t,x)$ is one periodic in t, say, we can also choose $y = (t,x) \in \mathcal{M} := \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. The tangent space becomes the domain of your new variable.

1.2 Scalar Equations: Solving 1st order autonomous ODEs

Autonomous: $\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}.$ (1.10)

Existence: In the instance $f(x_0) = 0,$ we simply set a $x(t) \equiv x_0$ to find an equilibrium solution.

For additional solutions, we first find solutions to an inverted version of the problem.

> **Inverse Function Theorem** (**IFT**): Gives a sufficient condition for a function F to be invertible in a neighborhood of a point x_0 in its domain: namely, that its derivative F' is continuous and non-zero at x_0. The theorem also gives a formula for the derivative of the inverse function.
> For functions of a single variable, the theorem states that if F is a continuously differentiable function with nonzero derivative at the point x_0, then F is invertible in a neighborhood of x_0, the inverse is continuously differentiable, and the derivative of the inverse function at $F_0 = F(x_0)$ is the reciprocal of the derivative of F at x_0: $\varphi'(F_0) := (F^{-1})'(F_0) = \frac{1}{F'}$.
> And so locally, we have a function $\varphi'(F) = \frac{1}{F'}$.

So, suppose $f(x_0) \neq 0$ (and C^0). Since $\dot{x} \neq 0,$ we can invert (IFT since $x \in C^1$) with $t = \varphi(x)$ locally and find $\varphi'(x) = \frac{1}{f(x)}$. Integrating, $\varphi(x(t)) = \int_{x_0}^{x(t)} \frac{1}{f(y)} dy.$ (1.11)

The right-hand side is a function of the form $T(x(t);x_0)$, which is monotone in $x(t)$ on intervals where the integrand does not change sign (away from zeros of f, monotone needed for IFT). We can therefore invert T and find a solution $x(t;x_0) = T^{-1}(t;x_0)$.

1.2.2 Uniqueness

We've found solutions for all init. conditions. For smooth enough f, those solutions are the only solutions. So, we say that $x(t)$ is a solution to (1.10) if $x(t)$ satisfies the equation and is differentiable on some open interval. Note that, for continuous f, $x(t)$ is in fact continuously differentiable since its derivative $f(x(t))$ is continuous.

Proposition 1.1. Assume $f \in C^1(\mathbb{R};\mathbb{R})$. Then (1.10) possesses unique solutions in the following sense.

(i) For all x_0, there exists $\delta > 0$ and a solution $x(t)$ for $|t| < \delta$.

(ii) Any two solutions coincide on their common domain of definition.

(iii) Solutions are given as $x(t) \equiv x_0$ when $f(x_0) = 0$ and implicitly through (1.11) otherwise.

Remark 1.2 (Uniqueness). It is sufficient to assume f to be locally Lipshitz (weaker than C^1, $C^0 \Rightarrow LL$); see the next chapter on uniqueness. Yet weaker assumptions are indeed sufficient and known, for instance Osgood's criterion. Note also (from the proof in the notes) that uniqueness is difficult to guarantee only near equilibria.

1.2.3 Nonuniqueness

A simple example with non-uniqueness involves power-laws: $x' = |x|^\beta$, which is not differentiable for $\beta < 1$ (for $\beta = 1$, it is Lipshitz and solutions are unique). For $x_0 = 0$, one finds the nontrivial solution: $\frac{1}{x}dx = t + c \quad \Rightarrow \quad \frac{x^{1-\beta}}{1-\beta} = t + c$. For $x \neq 0,$ we can define a piecewise definition of our solution: $x(t) = (1-\beta)^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \quad t \geq 0,$ and $x(t) = 0$ for $t \leq 0$. $x_0 \equiv 0$ also a solution. So no uniqueness for solutions through $x = 0$.

By altering the equations in our piecewise definition such that $t \leq / \geq c$ for some $c > 0$, we still find this to be a solution with $x_0 = 0$, but this shows more nonunique solutions! Indeed, we have a continuous family of nontrivial solutions. Note that all solutions are continuously differentiable at the piecewise junction since $\beta < 1$ and therefore $\frac{1}{1-\beta} > 1$.

1.2.5 First Integrals and Hamiltonian Systems

For an equation $x' = f(x);\ x \in \mathbb{R}^n$, we say that: $I : \mathbb{R}^n \to \mathbb{R}$, is a first integral if $I(x(t)) \equiv$ const; independent of time for all solutions. Equivalently, $0 = \frac{d}{dt} I(x(t)) = \left\langle \nabla I(x(t)), f(x(t)) \right\rangle_{\mathbb{R}^n}$. That is, $\nabla I(x) \perp f(x)$ for all $x \in \mathbb{R}^n$. The first integral is called non-trivial if I itself is not constant. It is non-degenerate if $\nabla I \neq \vec{0}$.

Locally (and sometimes globally), a non-degenerate first integral allows one to express one of the variables in terms of the others. If, for instance, $\partial_{x_1} I \neq 0$ (for instance, it is equal to the angular momentum), we can solve the equation $I(x) = I(x_0) = const$ explicitly in terms of one of the system variables: $x_1 = g(x_2, \dots, x_n; I(x_0))$, and substituting this in, obtain a differential equation for $y = (x_2, \dots, x_n)$, only. Of course, this new equation could again have a first integral, so we continue the process until we complete all the reductions.

First integrals are therefore useful, but usually hard to come by. We saw a simple example in the mass-action kinetics and the opinion model, where total numbers of atoms (or agents) were conserved.

An exception are Hamiltonian systems, such as the nonlinear pendulum equations discussed earlier. Somewhat more formally, we can consider a **Hamiltonian function** H and a symplectic matrix J, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \text{ with Hamiltonian equation } u' = J\nabla H(u). \qquad (1.12)$$

For the pendulum, $n = 1$ and $H(\varphi, v) = \frac{1}{2}v^2 - \frac{g}{\ell}\cos\varphi$. For the gravitational system, $H(x, v) = \sum_j \frac{1}{2} m_j |v^j|^2 + V(x)$; and the gradient is taken with respect to the mass-weighted scalar product. One often writes $u = (p; q)$ for p momentum and q position, and studies (after scaling) Hamiltonians of the form $H(p; q) = \frac{1}{2m}|p|^2 + V(q)$, referring to kinetic energy $T = \frac{1}{2m}|p|^2$ and potential energy $V(q)$. The total energy H is then a first integral (below we use skew-symmetry $J = -J^T$, and the fact that $\langle x, y\rangle = \langle y, x\rangle$ on $\mathbb{R}$):

$$(\nabla H, f) = (\nabla H, J\nabla H) = (J^T\nabla H, \nabla H) = (-J\nabla H, \nabla H) = (\nabla H, -J\nabla H).$$

So, $J\nabla H = -J\nabla H$, or $J\nabla H = 0$, and $(\nabla H, f) = 0$.

Proposition 1.3. Let $f \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ be a vector field with $div(f) = 0$. Then there exists $H \in C^2$ such that: $f = J\nabla H$.

Note that $\operatorname{div}(f) = \operatorname{div}\left(\dot{q}, \dot{p}\right) = \operatorname{div}(\partial_p H, -\partial_q H) = \partial_q\partial_p H - \partial_p\partial_q H = 0$.

Proof. We claim that $J^{-1}f$ is a gradient. Recall $\operatorname{curl} g = 0 \Rightarrow \exists \nabla$ where $g = \nabla G$. So our result follows immediately from the fact that the rotation/curl of $J^{-1}f$ is simply given by the divergence of f and hence vanishes (the rotation/curl of $J^{-1}f$ where $\operatorname{div} f = 0$ is $\operatorname{div} f = 0$).

2 Flows and Vector Fields

We always fix x a phase space: $X = \mathbb{R}^n$; or Banach space; or manifold (or open subset of …); and think of points $u \in X$ as the state of the system. Pairs $(u; t) \in X \times \mathbb{R}$ belong to the extended phase space.

Definition 2.1 (**Flow**). A differentiable map $\Phi : \mathbb{R} \times X \to X,\ (t, u) \to \Phi(t, u) =: \Phi_t(u)$; is called a flow if it satisfies the **cocycle property**: $\Phi_0 = id,\quad \Phi_t \circ \Phi_s = \Phi_{t+s}$, for all $t, s \in \mathbb{R}$. In particular, Φ_t is invertible with inverse Φ_{-t}. One mostly asks for continuity or even smoothness. The flow is continuous if Φ_t is continuous, hence Φ_t is a homeomorphism for all t. The flow is of class C^k if Φ is of class C^k, hence Φ_t is a C^k-diffeomorphism for all t. We will assume that $k > 1$ unless explicitly stated otherwise.

Definition 2.2 (**Local flows and semi flows**). A semi-flow is defined for $t > 0$, only, and the cocycle property holds for $t; s > 0$, only. For a local flow, Φ is defined in an neighborhood of $\{0\} \times X$ and the cocycle property holds when both left and right-hand side are defined.

Lemma 2.6. Let f be the vector field associated with the flow Φ_t. Then $x(t) := \Phi_t(x_0)$ solves $\dot{x}(t) = f(x(t))$.

Proof. We compute $x'(t_0) = \frac{d}{dt}|_{t=t_0}\Phi_t(x_0) = \frac{d}{dt}|_{t=t_0}\Phi_{t-t_0}(\Phi_{t_0}(x_0)) = \frac{d}{d\tau}|_{\tau=0}\Phi_\tau(\Phi_{t_0}(x_0)) = f(\Phi_{t_0}(x_0)) = f(x(t_0))$.

2.2 Banach's Fixed Point Theorem

2.2.1 Functional analysis basics

Metric Space: A set X equipped with a metric $d : X \times X \to \mathbb{R}_+$, $d(x,y) = d(y,x)$, $d(x,z) \le d(x,y) + d(y,z)$, $d(x,y) = 0$ iff $x = y$. The metric defines a topology and convergence, $x_n \to y$ iff $d(x_n, y) \to 0$.

Complete Metric Space: A metric space is called complete if Cauchy sequences converge, that is:

$$\lim_{N\to\infty} [\sup_{m,n\ge N} d(x_n, x_m)] = 0 \Rightarrow \exists y \text{ such that } x_n \to y.$$

An odd example is the discrete metric, $d(x;y) = 1$ unless $x = y$, more interesting examples are geodesic distances on manifolds. The most common case is $|x - y|$ in a subset of $\mathbb{R}^n$.

Normed Vector Space: Vector space X with a norm operator $|\bullet| : X \to \mathbb{R}_+$, where $|\lambda x| = |\lambda||x|$, $|x + y| \le |x| + |y|$, $|x| = 0$ iff $x = 0$. A normed space is a metric space with distance $d(x,y) = |x - y|$.

A complete normed space is a **Banach space**. Any closed subset of a Banach space is a complete Banach metric space.

A map $F : X \to Y$ between metric spaces is **Lipshitz continuous** with Lipshitz constant L if $d_Y(F(x_1), F(x_2)) \le L d_X(x_1, x_2)$, for all $x_1, x_2 \in X$.

The map is locally Lipshitz if for every $x \in X$ there exists a neighborhood $U(x)$ such that the restriction of F to $U(x)$ is Lipshitz continuous, with a Lipshitz constant $L(x)$. One readily shows that locally Lipshitz functions are Lipshitz on compact sets.

2.2.2 Banach's fixed point theorem

Contraction: A Lipshitz map $F : X \to X$ on a complete metric space with Lipshitz constant $\kappa := L < 1$.

Theorem 1 (**Banach's FP Theorem**). A contraction F possesses a unique fixed point (a point x where $F(x) = x$).

2.3 The Implicit Function Theorem

Equivalence of Boundedness and Continuity for Linear Operators L on Normed Spaces X and Y.

- Suppose that L is bounded. Then, for all vectors v and h in X with h nonzero we have: $|L(v + h) - L(v)| = |L(h)| \le M|h|$. Letting $h \to 0$ shows that L is continuous at v. Moreover, since the constant M does not depend on v, this shows that in fact L is uniformly continuous, and even Lipschitz continuous.
- Conversely, it follows from the continuity at the zero vector that there exists a $\delta > 0$ such that: $|L(h)| = |L(h) - L(0)| \le 1$ for all vectors h in X with $|h| \le \delta$. Thus, for all nonzero v in X, one has: $|Lv| = \left| \frac{|v|}{\delta} L\left(\delta \frac{v}{|v|}\right) \right| = \frac{|v|}{\delta} \left| L\left(\delta \frac{v}{|v|}\right) \right| \le \frac{|v|}{\delta} \cdot 1 = \frac{1}{\delta}|v|$, which proves that L is bounded.

Given $U \subset X, V \subset Y$, open subsets of Banach spaces, we consider differentiability of maps $F : U \to V$. We equip the vector space of bounded linear operators $\mathcal{L}(X,Y)$ with the norm: $|L| = \sup_{x\ne 0} \frac{|Lx|}{|x|}$, which makes it a Banach space. We say F is (**Frechet**) **differentiable** in x if there exists a bounded linear operator $L(x)$ such that for all $u \ne x \in U$,

$F(u) - F(x) - L(x) \cdot (u - x) = o(x - u)$, where the Landau symbol $o(\delta)$ satisfies $\frac{o(\delta)}{|\delta|} \to 0$ for $\delta \ne 0$. We call $L(x)$ the derivative at x and write: $DF(x) := L(x)$.

Higher derivatives are defined accordingly as derivatives of $DF(x) : X \to \mathcal{L}(X,Y)$ and so forth. They can be viewed as multi-linear maps $D^k F(x) : X^k \to Y$. We have the usual theorems from calculus such as the chain rule, differentiability of bounded multilinear maps (multivariate polynomials), continuous differentiability based on continuous partial derivatives, and Taylor's theorem. The inverse and implicit function theorem hold. We write C^∞ for functions with infinitely many derivatives and C^ω for (analytic) functions with convergent Taylor series.

A useful characterization of differentiability is based on directional derivatives. We say $F : X \to Y$ is **Gateaux differentiable** in x if for any $x_0 \in X$ the function $f_x : \mathbb{R} \to Y$, $f_x(t; x_0) = F(x + tx_0)$ is differentiable in t at $t = 0$, $\forall x$. Note that Frechet $\Rightarrow$ Gateaux.

Proposition 2.8 (**Gateaux $\Rightarrow$ Frechet**). Suppose F is Gateaux differentiable and $f_x(t; x_0) = F(x + tx_0) = L(x)x_0$ for some bounded operator $L_t(x)$ that depends continuously on x. Then F is Frechet differentiable.

Theorem 2 (**Implicit Function Theorem**). Suppose $F \in C^k(X \times \Lambda, Y)$, $k > 1$, $F(x_0; \lambda_0) = 0$, and $D_x F(x_0; \lambda_0)$ is

invertible with bounded inverse. Then there exists a neighborhood U of λ_0 and a neighborhood V of x_0, such that there is a unique solution to $F(x;\lambda) = 0$ for all $\lambda \in U$ with $x \in V$, given through $x = \varphi(\lambda)$. Moreover, $\varphi \in C^k$, $\varphi(\lambda_0) = x_0,\ D_\lambda\varphi = -\partial_x F^{-1}\partial_\lambda F$ at $x = x_0,\ \lambda = \lambda_0$.

Theorem 3 (**Banach's Fixed Point Theorem with parameters**). Suppose $F \in C^k(U_X \times U_\Lambda, U_X)$, defined on open subsets U_X and U_Λ of Banach spaces X and Λ, $k > 1$, F is a contraction in its first argument uniformly with respect to the second, $|F(x_1,\lambda) - F(x_2,\lambda)| \le \kappa|x_1 - x_2|$, for some $\kappa < 1$, and all x_1, x_2, λ. Then the unique fixed point $x_*(\lambda)$ is a C^k-function of λ.

2.4 Existence and Uniqueness

Recall the definition of locally Lipshitz functions. We say that a function $x(t)$, where $t \in J$ an interval, solves the differential equation (is a solution of) $x'(t) = f(x(t))$ for some continuous f if $x(t)$ is continuously differentiable and solves the differential equation.

Theorem 4 (**Picard-Lindelof existence and uniqueness**). Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ a locally Lipshitz vector field on U. Then for all $x_0 \in U$ there **exists** $t_b(x_0) > 0$ and a solution $x : [-t_b; t_b] \to U$ of $x' = f(x);\quad x(0) = x_0$: Moreover, given any solution $\tilde{x}(t)$, where $t \in J$ an interval, and $\tilde{x}(0) = x_0$, then $\tilde{x}(t) = x(t)$ for all $t \in J \cap [-t_b; t_b]$ (we have **uniqueness**). The proof steps consist of: Preconditioning, Localization, Picard Iteration, Well-defined Iteration, Contraction Property, Uniqueness-Restriction, and Uniqueness.

2.8 Dependence on Parameters – Proofs

Instead of the normal method for finding solutions, we can instead take a partial derivative of our given ODE with respect to some parameter μ, solve the resulting differential equation, then integrate the solution with respect to μ. The results will be a solution to the original ODE. We consider as always the initial-value problem: $x' = f(x,\mu),\ x(t_0) = x_0,$ (2.8)

with unique, local-in-time (t near t_0) solution $x(t; t_0, x_0, \mu)$.

Theorem 5 (**Smooth dependence on parameters**). Suppose $f \in C^k(U \times P, \mathbb{R}^n)$, $U \times P \subset \mathbb{R}^n \times \mathbb{R}^p$, for some $1 \le k \le \infty, \omega$. Then the unique solution $x(t; t_0, x_0, \mu)$ and its derivative with respect to time t are C^k in all variables. "Smooth f gives smooth x and x'."

Lemma 2.10. let $J \subset \mathbb{R}$ be an interval and suppose that $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$. Then $F(x)(t) := f(x(t))$ with $F : C^0(J, \mathbb{R}^n) \to C^0(J; \mathbb{R}^m)$ is of class C^k, for any $k = 1, 2, \ldots, \infty, \omega$. "Give me a C^k vector field, and I'll give you a C^k vector field along the related flow."

2.9 Other Existence and Uniqueness Results

Theorem 6 (**Peano's existence**). Suppose f is continuous at x_0. Then there exists $\delta > 0$ and a solution of the initial-value problem: $x' = f(x), \qquad x(0) = x_0,$ on $(-\delta, \delta)$. But it's not necessarily unique.

Proof:

Euler's approach. We construct guesses for solutions using Euler's iteration $x_{n+1} = x_n + h_k f(x_n)$, with

$h_k = \frac{1}{k}\delta, k,\ n = 0, \ldots, k-1$. From the points x_n we construct an approximate solution using linear interpolation.

Since the slopes of this interpolation are bounded by the vector field, we find an equi-continuous family of approximate solutions which by Arzela-Ascoli possesses a convergent subsequence. One then verifies that the limit is a solution to the differential equation using continuity of the vector field.

2.10 Local and Global Flows

(Finally, we have a flow!!) Smooth dependence on parameters gives existence of a local flow. Indeed, the map: $(t,x_0) \to x(t;x_0) =: \Phi_t(x_0)$ is defined in a neighborhood of $\{0\} \times \{x_0\} \subseteq \mathbb{R} \times \mathbb{R}^n$ and defines a C^k-map when $f \in C^k$ w/$k \geq 1$ or Lipshitz. Moreover, it has an inverse Φ_{-t}, by uniqueness of solutions, which is also differentiable. In particular, the derivative $\partial_{x_0}\Phi_t(x_0)$ is invertible and $\Phi_t(\cdot)$ is a local diffeomorphism onto its image. If the solution exists for all times $t \in \mathbb{R}$, the differential equation generates a flow in the neighborhood of $\{0\} \times \{x_0\}$.

2.11 Global solutions

We revisit the differential equation $x' = f(x)$. Let x_1 and x_2 be solutions on intervals T_1 and T_2, respectively, and suppose that for some $t_0 \in T_1 \cap T_2$ we have $x_1(t_0) = x_2(t_0)$. Then $x_1(t) = x_2(t)$ on $T_1 \cap T_2$, since the interval of coincidence $I = \left\{t \mid x_1(t) = x_2(t)\right\} \subset T_1 \cap T_2$ is nonempty by assumption, closed by continuity, and open by local uniqueness of solutions. We may therefore define a maximal interval of existence of solutions to the initial-value problem $x(0) = x_0$: $T_{max} = \bigcup_{\exists \text{ sol on } 0 \in T} T$.

Proposition 2.11 (**Sol**. **Blowup criteria**). Let f be locally Lipshitz, defined on $U \subset \mathbb{R}^n$. The maximal time interval of existence T_{max} is nonempty and open. If $T_{max} \neq \mathbb{R}$ and $t_* \in \partial T_{max}$ with $t \to t_*$ in T_{max}, we have that either $x(t) \to \partial U$ (global on the set of definition) or $|x(t)| \to \infty$ (not globally Lipschitz).

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3 Numerical Methods

The order of a numerical method. Let Φ_t be the flow to an ODE $x' = f(x)$, where we assume that f is a smooth function. We denote by φ_h a numerical approximation to the solution at time h. Since $x(h) = x_0 + \int_0^h f(x(s))ds$ [1]

we could choose for instance: $x(h) \approx \varphi_h(x_0) := x_0 + hf(x_0)$; approximating the integrand by a constant function with constant slope $f(x_0)$ derived from the left end of the domain of integration. This approximation is known as the explicit Euler method. Approximating the integrand by its right value gives the implicit form: $x(h) \approx \varphi_h(x_0) := x_0 + hf(\varphi_h(x_0))$; known as the implicit Euler method. Since Φ_h is smooth in h, we can expand the solution (and the integral) in h and find at leading order $\Phi_h(x_0) = x_0 + hf(x_0) + O(h^2)$. We say that a numerical method is of order p if the **local error** is of order $p+1$, $\Phi_h(x_0) - \varphi_h(x_0) = O(h^{p+1})$, (as h increases, the error grows as h^{p+1}).

The **global error** is the accumulated error over all solution steps. We define the global error at step n in any numerical approximation where $h = \frac{t}{n}$, the global error at a fixed time t : $E_n = |\varphi_h^n(x_0) - \Phi_{nh}(x_0)| \leq nO(h^{p+1}) = O(h^p)$ is of order p. Rigorous statements assume for instance that f is globally Lipshitz with constant L and find error estimates $\left|\varphi_h^{\frac{t}{h}}(x_0) - \Phi_t(x_0)\right| \leq Ce^{Lt}h^p$.

Backward Error Analysis. An intriguing point of view asks if our numerical scheme, although not solving the differential equation exactly, it exactly solves a slightly different equation $x' = f_h(x), \quad x(0) = x_0$; for a suitable f_h. This is usually true up to much smaller errors, $O(e^{-\frac{c}{h}})$ ($\leq Che^{-\frac{c}{h}}$) for analytic vector fields f. Similarly, one can ask if we are solving exactly for a slightly different *initial condition*, $x' = f(x), \quad x(0) = \tilde{x}_0(h)$; thus improving the global-in-time bound. This again turns out to be true under reasonable conditions, that are however hard to guarantee a priori. These conditions of uniform hyperbolicity then give uniform-in time error bounds $O(h^p)$.

Runge-Kutta. One would like to evaluate the integral $\int_0^h f(x(s))ds$ using a higher-order integration formula, usually based on polynomial interpolation at points $t_k \in [0,h]$ for this, we need to construct approximations $f_k \approx f(x(t_k))$. A simplest (first order) formula would use the trapezoidal rule: $\int_0^h f(x(s))ds \approx \frac{h}{2}(f(x_0) + f(\varphi_h(x_0)))$, which gives the implicit expression for $x_h = \varphi_h(x_0)$ as $x_h - x_0 = \frac{h}{2}(f(x_0) + f(x_h))$. For h small, this equation can be quickly solved using few a Newton steps and an initial guess $x_h^0 = x_0$.

For higher order, one computes $x_k \sim x(t_k)$ using: $x_k = x_0 + \int_0^h f(x(s))ds$. The integral is approximated by $\int_0^h f(x(s))ds \sim h\sum_{i=1}^s b_i f_i$, where $f_i = f(x_i) = f(0 + h(\Sigma_{k<i} a_{ij} f_i)) = \ldots$ and a_{ij}, b_i is taken from a "butcher tableau."

Backward Differentiation Methods: These are numerical integration methods based on Backward Differentiation Formulas (BDFs). They are particularly useful for stiff differential equations and Differential-Algebraic Equations (DAEs). BDFs are formulas that give an approximation to a derivative of a variable y at a time t_n in terms of its function values $y(t)$ at t_n and earlier times (hence the "backward" in the name). They are derived by forming the k-th degree interpolating polynomial approximating the function $y(t)$ using $y(t_n), y(t_{n-1}), \cdots, y(t_{n-k})$, differentiating it, and evaluating it at t_n. For example, the linear interpolating polynomial through $y(t_n)$ and $y(t_{n-1})$ is: $y(t) \approx y(t_n) + (t - t_n)\frac{y_n - y_{n-1}}{t_n - t_{n-1}}$. So the approximation to the derivative is the familiar: $y_n' \approx \frac{y_n - y_{n-1}}{t_n - t_{n-1}}$. If this is used to obtain a numerical approximation to the ordinary differential equation $y' = f(y,t)$, by replacing the derivative on the left hand side of this DEQ, one obtains the Backward Euler method: $y_n = y_{n-1} + (t_n - t_{n-1})f(y_n, t_n)$. (2)

If y_{n-1} is known, then equation (2) is implicit in y_n (implicitness is essential for arbitrarily stiff systems). Because equation (2) is based on a linear approximation to $y(t)$, it is a first-order method.

Backward Difference Operator ∇: $\nabla y_n = y_n - y_{n-1}$, $\nabla^p y_n = \nabla^{p-1} y_n - \nabla^{p-1} y_{n-1}$ (for example: $\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$).

The k-th degree polynomial interpolating $y_n, y_{n-1}, \cdots, y_{n-k}$ can be written using backward differences as:

$$y(t) \approx y_n + \frac{1}{h}(t - t_n)\nabla y_n + \frac{1}{2h^2}(t - t_n)(t - t_{n-1})\nabla^2 y_n + \ldots + \frac{1}{h^k k!}(t - t_n)\ldots(t - t_{n-(k-1)})\nabla^k y_n. \quad (5)$$

If equation (5) is differentiated and t is set to t_n, the k-th order BDF is obtained. Formula has form: $hy_n' = \sum_{j=0}^k \alpha_{kj} y_{n-j}$.

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4. Qualitative Dynamics

Proposition 4.1 (**Change of Coordinate**): Let ψ be a smooth diffeomorphism on $\mathbb{R}^n$ and Φ_t be the flow to $x' = f(x)$. Then, $\widetilde{\Phi}_t := \psi^{-1} \circ \Phi_t \circ \psi$ is a flow, for the variable $y = \psi^{-1}(x)$, and ODE $y' = (D\psi(y))^{-1} f(\psi(y))$. (Derived by: $x = \psi(y) \Rightarrow x' = D\psi(y) \bullet y'$)

Remark 4.2. The diffeomorphism should be thought of as a change of coordinates. Also, note that a C^1-diffeomorphism ψ gives a continuous vector field, only, since $D\psi$ is only continuous. One therefore finds an ODE that would not necessarily have unique solutions, even though we have a well-defined flow associated to it. In a more extreme direction, we could only require that ψ be a homeomorphism and obtain a continuous flow $\widetilde{\Phi}_t$.

Hartman–Grobman Theorem (**hyperbolic linearization theorem**): Deals with the local behavior of dynamical systems in the neighborhood of a hyperbolic equilibrium point. It asserts that linearization is effective in predicting qualitative patterns of behavior. The theorem states that the behavior of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point, where hyperbolicity means that no eigenvalue of the linearization has real part equal to zero. Therefore, when dealing with such dynamical systems one can use the simpler linearization of the system to analyze its behavior around equilibria.

Consider a system evolving in time with state $x(t) \in \mathbb{R}^n$ that satisfies the differential equation $\dot{x} = f(x)$ for some C^1 map $f: \mathbb{R}^n \to \mathbb{R}^n$. Suppose the map has a hyperbolic equilibrium state $x_0 \in \mathbb{R}^n$: that is, $f(x_0) = 0$ and the Jacobian matrix $J = [\frac{\partial f_i}{\partial x_j}]$ of f at state x_0 has no eigenvalue with real part equal to zero. Then there exists a neighborhood N of the equilibrium x_0 and a homeomorphism $h : N \to \mathbb{R}^n$, such that $h(x_0) = 0$ and such that in the neighborhood $N, \exists t_0 > 0$ s.t. $\forall t \in (-t_0, t_0)$ and $x \in N$ we have $h \circ \varphi_t(x) = e^{At} h(x)$, i.e. flow is topologically conjugate to flow of linearization. In other words: the flow of $\dot{x} = f(x)$ is topologically conjugate by the continuous map $U = h(x)$ to the flow of $dU/dt = JU$.

Blowing-Up Techniques on $\mathbb{R}^2$ for non-hyperbolic f.p.s: Involve changes of coordinates which expand, or "blowup," the non-hyperbolic fixed point (assumed to be at $x = 0$) into a curve on which a number of singularities occur. The topological type of each of these singularities is then investigated using the Hartman Theorem. The changes of coordinates used are, of course, singular at the fixed point, since they map a curve onto a point. Elsewhere, however, they are diffeomorphisms. The simplest example is well known to everyone, namely plain polar coordinates.

- First, perform the blowup with polar coordinates
- Then, perform stereographic projection
- Next, use algebra to analyze

4.2 Time Rescaling (with diffeomorphism $\gamma(t)$)

Proposition 4.3: Suppose $\alpha : \mathbb{R}^n \to \mathbb{R}^+$ is smooth, strictly positive. Then trajectories $\{x(t), t \in \mathbb{R}\}$ to $x' = f(x)$ are trajectories to $y' = \alpha(y)f(y)$. More precisely, there is $\gamma : T \to T'$, a diffeomorphism of the maximal existence time intervals for $x' = f(x)$ and $y' = \alpha(y)f(y)$ with initial condition $x(0) = y(0) = x_0$, respectively, such that $y(\gamma(t)) = x(t)$.

Proof: We substitute $y(\gamma(t))$ into $x' = f(x)$ and find $\gamma'(t)\alpha(x(t))f(x(t)) = f(x(t))$.

This equation is clearly satisfied when $\gamma'(t) = (\alpha(y(\gamma(t))))^{-1}$. ■

Note that we can therefore first solve the (hopefully simpler) equation for y. Then, we determine γ through the solution of a simple scalar differential equation. And then obtain the function $x(t)$. The function α is sometimes called an **Euler multiplier**, or, in special cases, an integrating factor. Roughly speaking, we change the vector field by a factor depending on the point in phase space. The curves tangent to this vector field are unchanged by this "stretching" of the vector field. It is worth noticing that the vector field gives the tangent vector to solutions, not a "force" acting on a point in phase space. Changing the strength of the force (that is, having inertia), would of course change the trajectory.

Remark 4.4 (**Reversing time**). The case $\alpha < 0$ is equivalent, trajectories still coincide but are traveled in the opposite direction of time. Whenever α changes sign, one needs to separate regions where α is positive and where α is negative.

Scaling: A very important structure in differential equations are scaling symmetries. Consider the scale invariant equation (with a kind of spherical symmetry) $u' = f(u) \in \mathbb{R}^n$, $f(\lambda u) = \lambda^p f(u)$ for all $\lambda > 0$ (λ^p is an Euler multiplier). Note that $f(0) = 0$ by continuity as λ gets small ($0 \approx \lambda^p f(u) = f(\lambda u) \approx f(0)$). We change to polar coordinates, $u = Rv$, $R > 0$, $v \in S^{n-1} \subset \mathbb{R}^n$, $|v| = 1$. Since v always lives in the sphere, we can conclude $\langle v', v \rangle = 0$, which lets us solve for v'. Then, with substitution we can eliminate v' in our DEQ and reduce the dimensions of the DEQ by one.

4.3 The Flow Box

From a theoretical perspective, one would wish to describe "qualitative" features of differential equations, only. Qualitative here can be specified as information that does not depend on coordinate changes. One could then say that two flows are equivalent if there exists a coordinate change that conjugates one to the other, $\Phi_t = \psi^{-1} \circ \widetilde{\Phi}_t \circ \psi$.

Proposition 4.5 (**Flow-Box**): Let $f \in C^k$ w/$k \geq 1$, and $f(x_0) \neq 0$. Then, there exists a local diffeomorphism $\psi \in C^k$ from a neighborhood of x_0 to a neighborhood N of the origin, such that $y := \psi(x)$ satisfies $y' = f(x_0) = const$, $\forall y \in N$. In other words, we can locally straighten flowlines and achieve constant speed.

4.4 Invariant Sets and Limit Sets

We suppose throughout that we have a flow Φ_t on a phase space X, thinking of $X = \mathbb{R}^n$.

We say a set $S \subset X$ is :

- Invariant, if $\Phi_t(S) \subset S$ for all $t \in \mathbb{R}$,
- Forward Invariant, if $\Phi_t(S) \subset S$ for all $t \geq 0$,
- Backward Invariant, if $\Phi_t(S) \subset S$ for all $t \leq 0$.

Invariant sets are unions of trajectories. Level sets of first integrals are invariant.

The ω-limit set of a point (or of the trajectory through that point) is the set of accumulation points as time tends to infinity: $\omega(x_0) = \left\{ y \in X \mid \exists t_k \to \infty,\ \Phi_{t_k}(x_0) \to y \right\}$.

Examples:

- x_0 equilibrium $\Rightarrow \quad \omega(x_0) = \alpha(x_0) = \{x_0\}$;
- x_0 on periodic orbit $\Rightarrow \quad \omega(x_0) = \alpha(x_0) = \gamma(x_0) := \{\Phi_t(x_0) | t \in \mathbb{R}\}$;
- x_0 homoclinic if $\omega(x_0) = \alpha(x_0) = \{x_*\},\ x_* \neq x_0$;
- x_0 heteroclinic if $\omega(x_0) = \{x_+\} \neq \alpha(x_0) = \{x_-\}$;
- $x' = 1$ w/$x \in \mathbb{R} \quad \Rightarrow \quad \omega(x_0) = \emptyset$.
- $x_1' = 1,\ x_2' = \beta,\ x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,\ \beta \notin \mathbb{Q} \quad \Rightarrow \quad \omega(x_0) = \alpha(x_0) = \mathbb{T}^2$ for all x_0.

Proposition 4.6 (**Topology of ω-limit sets**): Suppose that the forward orbit of x_0 is bounded. Then, $\omega(x_0)$ is:

i) Nonempty, ii) Compact iii) Connected, iv) Invariant, v) $\omega(x_0) = \cap_{T \geq 0} \overline{\cup_{t \geq T} \Phi_t(x_0)}$, and

$\lim_{t \to \infty} dist(\Phi_t(x_0), \omega(x_0)) = 0$, where $dist(y, A) = \inf_{z \in A} |z - y|$.

We can similarly define $\omega(U)$ for sets $U \subset X$, $\omega(U) = \cap_{T \geq 0} \overline{\cup_{t \geq T} \Phi_t(U)}$. Note in general, $\omega(U) \neq \cup_{x_0 \in U} \omega(x_0)$.

Counter Example

Consider the flow on $[0,1]$ to $x' = x - x^2$, where of course, $\omega([0,1]) = [0,1]$, but $\omega(x) = 1$ for all $x > 0$ and $\omega(0) = 0$ such that $\cup_x \omega(x) = \{0,1\}$.

Non-Wandering Set $\Omega : x \in \Omega \Leftrightarrow$ for all neighborhoods $U(x)$ there exists $t_k \to \infty$ such that $\Phi_{t_k}(U(x)) \cap U(x) \neq \emptyset$, and the **chain recurrent set CR**, $x \in CR \Leftrightarrow \forall \varepsilon, T > 0,\ \exists \varepsilon$-pseudo-orbit with $x_n = x$ (endpoint = start point), where ε-pseudo-orbits are piecewise orbits with at most ε-jumps, that is, there exists $T_j > T,\ x_j,\ 0 \leq j \leq n-1,\ |\Phi_{T_j}(x_j) - x_{j+1}| < \varepsilon$.

4.5 Stability

Consider the invariant set $\emptyset \neq M \subseteq X$. We say M is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\Phi_t(U_\delta(M)) \subset U_\varepsilon(M)$, for all $t \geq 0$. The set is asymptotically stable if it is stable and if, for all x_0 in some neighborhood $V(M)$, the $\lim_{t \to \infty} dist(\Phi_t(x_0), M) = 0$. Note that the definitions of invariant sets and stability are invariant under (homeomorphic) coordinate changes.

4.6 Lyapunov Functions

We say that $V : X \to \mathbb{R}$, continuous is a Lyapunov function if $V(\Phi_t(x))$ is non-increasing in time for all x. If $V \in C^1$, we find the equivalent condition: $\frac{d}{dt} V(\Phi_t(x)) \leq 0$, equivalently $(\nabla V, f) \leq 0$.

For strict Lyapunov functions, V is strictly decreasing along trajectories, unless x is an equilibrium.

Note: Sublevel sets $V_c = \left\{x \mid V(x) \leq c\right\}$ are forward invariant. If $V(x(t_*)) \leq c$ for some t_*, then for all $t > t_*$, we still have $V(x(t)) \leq c$.

Examples?

Proposition 4.7 (**LaSalle's invariance principle**):

i) Suppose V is a Lyapunov function, $\omega(x_0) \neq \emptyset$. Then V is *constant* on $\omega(x_0)$, that is, $V(\omega(x_0)) = \{V_0\}$.

ii) Suppose V is a strict Lyapunov. Then $y' = 0,\ \forall y \in \omega(x_0)$, that is, the ω-limit set consists of equilibria.

Corollary (**to Morse Lemma**): Consider $x' = f(x)$ with $f(0) = 0$, and assume that $V(x)$ is a Lyapunov function near 0 with $V(0) = 0,\ \nabla V(0) = 0$, and $D^2 V(0) > 0$. Then 0 is stable.

For linear systems $x' = Ax$, that we shall study later, we can find Lyapunov functions quite explicitly.

For $x' = Ax$, & neg definite $A = A^T$ (So $\sigma(A) < 0$) we've Lyap $V(x) = -\frac{1}{2}\langle Ax, x\rangle$. $\frac{d}{dt}V(x(t)) = -\frac{1}{2}\left(\langle A(Ax), x\rangle + \langle Ax, Ax\rangle\right)$ $= -|Ax|^2 < 0$. Also: $D^2V(x) = -\frac{1}{2}D^2(x^TAx) = -\frac{1}{2}D\left[(Ax + x^TA)\cdot\vec{1}\right] = -\Sigma A_{ij} > 0$ (neg def), $\nabla V(x)|_0 = -\frac{1}{2}(x^T(\nabla Ax) + (\nabla x^T)Ax)|_0 = -(x^TA + Ax)|_0 = 0$. And $f(0) = A(0) = 0$. So stable by Morse.

Negative definite tells us the eigenvalues are negative, and the Morse Lemma construction above yields stability. This argument generalizes quite readily to nonlinear problems as we shall see later. More generally, let's motivate the next chapter, in which we shall study linear equations. We saw from smooth dependence on initial conditions that we may approximate the solutions in the vicinity of a known solution by the linearized differential equation $x = x_0 + hF(x_0) + O(h^2)$. This approximation is however only valid for finite times.

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5 Linear Equations

5.1 Constant Coefficient Linear Equations

Consider $x' = Ax \in \mathbb{R}^n$, $x(0) = x_0$, $A \in \mathbb{R}^{n\times n}$. The matrix exponential $e^{At} := \sum_{k=0}^{\infty}\frac{(At)^k}{k!}$, is well defined since the sum on the right-hand side is absolutely convergent. It is in fact an analytic function of t. We have: $|e^{At}| \le e^{|A||t|}$, where $|A|$ denotes the matrix norm $|A| = \sup_{|x|=1}|Ax|$ relative to some norm $|\cdot|$ in $\mathbb{R}^n$. Because of absolute convergence, we may differentiate termwise and find $\frac{d}{dt}e^{At} = Ae^{At}$. As a consequence, $e^{At}x_0$ is the unique solution to $x' = Ax$.

Coordinate Changes: We consider linear coordinate changes $S \in GL(X)$, $x = Sy$, which gives: $y' = S^{-1}ASy$, with solutions $y(t) = e^{S^{-1}ASt}y_0$. Note that $e^{S^{-1}AS} = S^{-1}e^{At}S$, by the change of coordinates rule (or by inspecting the cancellations in the infinite sum). We therefore wish to compute e^{At} in the simplest set of coordinates possible.

Commuting Matrices: If $AB = BA$, then $e^{A+B} = e^Ae^B$, as a simple inspection of the Taylor series shows.

Lemma 5.1: Suppose $V \subset X$ is a subspace invariant under A, $AV \subset V$. Then V is invariant under the flow $\Phi_t = e^{At}$ (or $\Phi_t(x_0) = x_0e^{At}$ where $x_0 \in V$).

Linearity: Clearly, linear combinations of solutions are solutions, $\Sigma_j\alpha_jx_j(t)$ is a solution for any $\alpha_j \in \mathbb{R}$ if $x_j(t)$ are solutions. Therefore, if $X = V \oplus W$ (decomposes into invariant subspace is under A), it is sufficient to solve the differential equation in V and W, separately.

5.2 Jordan Normal Form

In $\mathbb{C}^n$, the real subspace and the imaginary subspace $V_{r,i}$ are invariant under a real square matrix A. As a consequence, changing coordinates $y = S^{-1}x \in \mathbb{C}^n$, the conjugate matrix $S^{-1}AS$ possesses (real linear) n-dimensional invariant subspaces $S^{-1}V_{r/i}$.

Note that algebraic and geometric multiplicities of eigenvalues are invariant under coordinate changes (although the related eigenvectors move), that is, they are the same for A and $S^{-1}AS$.

Diagonalizable Matrices: We say that an eigenvalue λ is **semi-simple** if $i_a = i_g$. If all eigenvalues are semi-simple, we say A is semi-simple. Semi-simple matrices are precisely the diagonalizable matrices.

Non-Semisimple Matrices: One finds a block diagonalization of A such that each block has only a single eigenvalue. If A_j is semi-simple, we can find an eigenbasis such that A_j is diagonal. One shows that $\mathbb{C}^n = \Sigma_{j=1}^{\ell}V_j$ (sum of eigenspaces) $AV_j \subseteq V_j$, and the restriction A_j of A to V_j (where $A_j := A : V_j \to \mathbb{C}^n$) possesses only a single eigenvalue λ_j. If A_j is non-semi-simple, we can reduce to a Jordan Normal Form (JNF).

Lemma 5.2: Suppose that the spectrum of $A \in \mathbb{C}^{n\times n}$ consists of $\lambda = 0$. Then there exists a basis of $\mathbb{C}^n$ such that $A_{ii} = 0$, $A_{i,i+1} \in \{0, 1\}$.

In practice, to find JNF, one first computes the kernel $u_1, \dots, u_\ell$ of A. If $\ell < n$, one tries to find generalized eigenvectors, solving $Av = \Sigma_{j=1}^{k}\alpha_ju_j$ for v and $(\alpha_j) \in \mathbb{C}^k$ in a fixed complement of the kernel. Note that $A^2v = 0$ for any such v.

The Jordan Normal Form: The generalization of Lemma 5.2 gives the following result. We define N_k as the matrix of dimension k with entries $N_{k,j,j+1} = 1$ for all $j = 1, \dots, n-1$, and $N_{k,j,\ell} = 0$, otherwise. For example: $N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Theorem 7 (**Jordan Normal Form**). For each matrix $A \in \mathbb{C}^n$, there exists a change of coordinates S such that $S^{-1}AS$ is block diagonal and each block is of the form $\lambda_j id_{k_j} + N_{k_j}$ (for example $\begin{bmatrix} \lambda_j & 1 & 0 \\ 0 & \lambda_j & 1 \\ 0 & 0 & \lambda_j \end{bmatrix}$), where $k_j \geq 1$ is the size of the block and λ_j is an eigenvalue.

Since algebraic and geometric multiplicities are invariant under coordinate changes we can read off multiplicities from the Jordan normal form. The geometric multiplicity of λ_j is given by the number of blocks with diagonal entries λ_j. Algebraic multiplicities are given by the sum of the lengths of the blocks with diagonal entry λ_j.

The Exponential in Jordan Normal Form: As we observed earlier, e^{At} leaves subspaces invariant that are invariant under A. Therefore e^{At} has the same block diagonal structure as A, when A is in Jordan Normal Form. It is therefore sufficient to compute the exponential of a single Jordan block, $\lambda id_k + N_k$. Since the identity and N_k commute, we have: $e^{(\lambda id_k + N_k)t} = e^{\lambda t} e^{N_k t} = e^{\lambda t} \sum_{j=0}^{k-1} \frac{t^j}{j!} N_k^j$, since $N^k = 0$. To see the latter, note that $Ne_j = e_{j-1}$, and $Ne_0 = 0$ for the canonical basis e_j of $\mathbb{R}^k$.

Ignoring all the computations, we can therefore find the general solution to $x' = Ax$ by substituting an ansatz: $x(t) = \sum_{\lambda \in \sigma(A)} p_\lambda(t) e^{\lambda t}$, into the equation, where $p_\lambda(t)$ is a vector valued polynomial in t with degree at most the maximal algebraic multiplicity of λ.

Perturbing JNF. The JNF is not only not unique, which is already inconvenient for computational purposes, but also discontinuous. Consider the family of matrices $\begin{pmatrix} \varepsilon & 1 \\ 0 & 0 \end{pmatrix}$. At $\varepsilon = 0$, it is in JNF, with trivial transformation $S = id$. For $\varepsilon > 0$, the JNF form is $\begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}$, and the eigenvectors $(1,0)^T$ and $(1,-\varepsilon)^T$ has a pole at $\varepsilon = 0$. The form takes all vectors to zero as $\varepsilon \to 0$, and the eigenvectors collide. Transformation Eigenvectors $S = \begin{pmatrix} 1 & 1 \\ -\varepsilon & 0 \end{pmatrix}$ have a pole at $\varepsilon = 0$.
$\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ 1 & \frac{1}{\varepsilon} \end{pmatrix} = SDS^{-1}$

Determinant and Trace: Note that both determinant and trace are products and sums of eigenvalues, that is, coefficients in the characteristic polynomial must be invariant under coordinate change, and can therefore be computed in JNF. We therefore have the following results:

Corollary 5.4: We have: $\det e^{At} = \prod_\lambda e^{\lambda t} = e^{\Sigma \lambda t} = e^{(trA)t}$.

It suffices to triangularize: $A = P^{-1}TP$, with P invertible and T upper-triangular. This is possible as soon as the characteristic polynomial splits, which is obviously true in $\mathbb{C}$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Observe that each T^k is upper triangular with $\lambda_1^k, \dots, \lambda_n^k$ on the diagonal. It follows that e^T is upper triangular with $e^{\lambda_1}, \dots, e^{\lambda_n}$ on the diagonal. So, $\det e^T = e^{\lambda_1} \dots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n} = e^{tr(T)}$.
Finally, observe that $tr(A) = tr(T)$, so $\det \mathbf{e}^{At} = |P^{-1}\mathbf{e}^{Tt}P| = |P^{-1}||\mathbf{e}^{Tt}||P| = \frac{|P|}{|P|} \prod_\lambda e^{\lambda t} = e^{\Sigma \lambda t} = e^{(trA)t}$.

Higher-Order Equations: Writing a higher order equation: $x^{(n)} + a_1 x^{(n-1)} + \ldots + a_{n-1} x' + a_n x = 0$, as a first order system, we end up with first order equations where $x' = Ax$, with $A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & a_{n-2} & \ldots & -a_1 \end{pmatrix}$, with characteristic polynomial $\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$, and eigenvectors $(1, \lambda, \lambda^2, \ldots, \lambda^n - 1)^T$, that is, the geometric multiplicity is always 1.

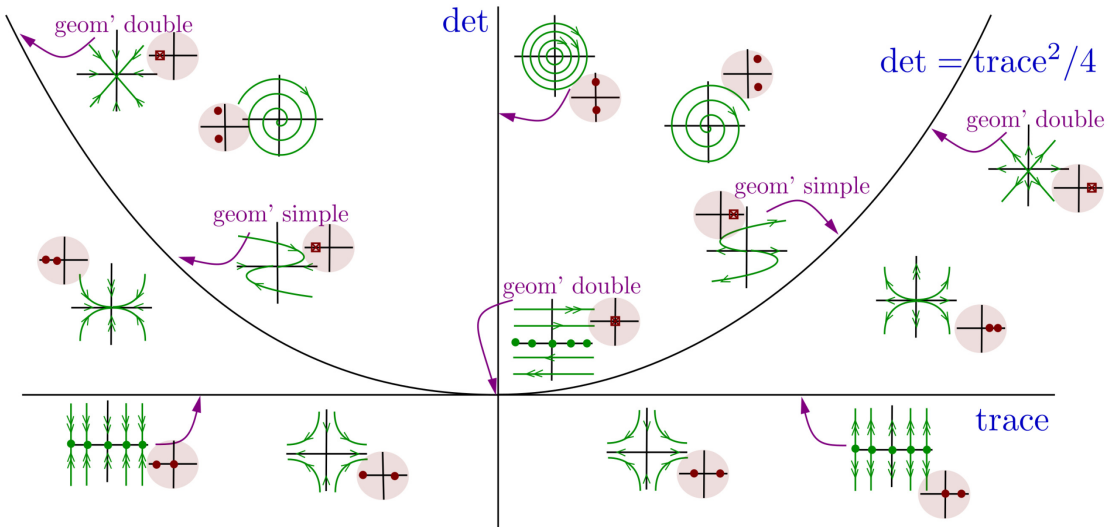
Spectral Projections: Projections are linear maps with $P^2 = P$, which implies that P is the identity on its range. Also, the range and kernel of P spans $\mathbb{R}^n$ (or $\mathbb{C}^n$). Particularly interesting are projections that commute w/ A, $PA = AP$ since such projections leave the range & kernel of A invariant under A. ($AP(range(A)) \subseteq P(range(A))$?)

Matrix Differential Equations: The DEQ $X' = AX$, where $A \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{n \times m}$, reduces simply to solving equations for all the columns x^j of the matrix X, simultaneously. If $m = n$, and $X(0) = id$, then $X(t) = e^{At}$ is called the fundamental matrix. If $X(t_0)$ is invertible, then the fundamental matrix can be calculated as $\Phi(t) = X(t)X^{-1}(t_0)$.

5.3 Planar Systems

We describe phase portraits of planar systems.

Eigenvalues are completely determined by trace and determinant, $\lambda_{1/2} = \frac{tr}{2} \pm \sqrt{\frac{tr^2}{4} - \det}$.



We can focus on $tr \geq 0$, since time reversal gives the case $tr < 0$.

Center: $tr = 0$, $\det > 0$. Eigenvalues $\lambda_{1/2} = \pm i\omega$ are purely imaginary and we find solutions that are ellipses or, in JNF, circles in the phase plane.

Unstable Focus: $tr > 0$, $\det > 0$, $\det > \frac{tr^2}{4}$. Eigenvalues are $\lambda_1, \bar{\lambda}_2 = \eta \pm i\omega$, with $\omega, \eta > 0$, solutions in the complex JNF $z = e^{i\omega t} e^{\eta t}$, hence logarithmic spirals $|z| \sim e^{arg(z)}$.

Resonant Node (**Twisted Star**): $tr > 0$, $\det > 0$, $\frac{tr^2}{4} = \det$. We have $\lambda_1 = \lambda_2 > 0$. If A is semi-simple, the solutions are simply $x(t) = x_0 e^{\lambda t}$ radially exponentially outward. If A is non-semi simple, we have in JNF $x_2 = e^{\lambda t} x_2^0$, $x_1 = te^{\lambda t} x_2^0 + e^{\lambda t} x_1^0$. Note that the eigenspace $x_2 = 0$ is invariant. Outside, one finds, for example when $\lambda = 1$, $x_2^0 > 0$, that $x_1 = x_2 \log x_2$, non-monotone profiles resembling both a logarithmic spiral and the straight node we will discuss next.

Unstable Node: $tr > 0,\ \det > 0,\ \det < \frac{tr^2}{4}$. Eigenvalues are real $\lambda_1 > \lambda_2 > 0$, solutions in JNF are $x_1(t) = x_1^0 e^{\lambda_1 t},\ x_2(t) = x_2^0 e^{\lambda_2 t}$, hence typical solutions are $x_1(t) \sim x_2(t)^{\frac{\lambda_1}{\lambda_2}}$ or parabolae.

Unstable/Zero: $tr > 0,\ \det = 0$. Eigenvalues are $\lambda_2 > \lambda_1 = 0$, solutions in JNF are $x_2(t) = x_2^0 e^{\lambda_2 t},\ x_1(t) = x_1^0$. Trajectories are parallel to the x_2-axis, converging to the x_1-axis in backward time.

Saddle: tr $> 0,\ \det < 0$. Eigenvalues are real $\lambda_1 > 0 > \lambda_2$, solutions in JNF are $x_1(t) = x_1^0 e^{\lambda_1 t},\ x_2(t) = x_2^0 e^{\lambda_2 t}$, hence typical solutions are hyperbolae: $x_1(t) \sim x_2(t)^{\frac{\lambda_1}{\lambda_2}}$. Solutions with $x_1^0 = 0$ converge to the origin in forward time, solutions with $x_2^0 = 0$ converge to the origin and backward time.

Zero Eigenvalues: $tr = 0,\ \det = 0$. If A semi-simple, $x' = 0$ & flow is trivial, all points are f.p.s. If A non-semisimple, we find in JNF: $x_2(t) = x_2^0,\ x_1(t) = tx_2^0 + x_1^0$, a shear flow parallel to x_1-axis w/f.p.s on x_1-axis.

5.4 The Adjoint Equation $x' = Ax$, and $\psi' = -A^*\psi$, (5.1)

where A^* is the adjoint with respect to a fixed scalar product $\langle \bullet, \bullet \rangle$ in $\mathbb{R}^n$.

Lemma 5.5: We have: $\frac{d}{dt}\langle (x(t), \psi(t)) \rangle = 0$, for any solutions $x(t), \psi(t)$ to (5.1).

Proof: Compute using the product rule:
$\frac{d}{dt}\langle x(t), \psi(t) \rangle = \left\langle \frac{d}{dt}x(t), \psi(t) \right\rangle + \left\langle x(t), \frac{d}{dt}\psi(t) \right\rangle = \langle Ax(t), \psi(t) \rangle + \langle x(t), -A^*\psi(t) \rangle = \langle x, A^*\psi \rangle + \langle x, -A^*\psi \rangle = 0.$
So we have a first integral, and can perform a reduction of our system!

One also observes that $PA = AP$ implies, taking adjoints, that $A^*P^* = P^*A^*$. In words, adjoints of spectral projections are spectral projections of adjoints.

Lemma 5.6: Let E_λ be the generalized eigenspace to the eigenvalue $\lambda \in \mathbb{R}$ of A and E_λ^c the sum of all other generalized eigenspaces. Similarly, define E_λ^* and $E^{c,*}$ using A^*. Then, $E_\lambda = (E_\lambda^{c,*})^\perp,\quad E_\lambda^c = (E_\lambda^*)^\perp$.

Non-Autonomous DEQ

5.5 Variation of Constant Formula (AKA Duhamel formula) for DEQ w/ Autonomous Homogeneous Part:

$x' = Ax + g(t),\quad x(0) = x_0$

Gives an explicit solution $x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(s)ds$. **Proof**: Differentiate!

5.6a Nonautonomous Homogeneous $x' = A(t)X,\quad x(t_0) = x_0$, for $A(t)$ continuous (piecewise continuous would be sufficient)

Remembering that derivatives $y(t) := \partial_{x_0}x(t;x_0)$ of solutions $x(t;x_0)$ to $x' = f(x),\ x(0) = x_0$ solve the linearized equations $y' = Df(x(t;x_0))y$, we wish to study the more general nonautonomous homogeneous linear equation above. We write $\Phi_{t,t_0}x_0$ for the solution, where $\Phi_{t_0,t_0} = id$, and Φ_{t,t_0} is linear, hence a $n \times n$-matrix in coordinates. We call $\Phi_{t,0}$ the fundamental solution. Since we can generate any other solution $\Phi_{t,s}$ as : $\Phi_{t,0} \bullet (\Phi_{s,0})^{-1}$. In the rare instance when $A(s)A(t) = A(t)A(s)$, we have $\Phi_{t,0} = e^{\int_0^t A(s)ds}$. Otherwise, below we examine one known solution in 5.7 when $A(t)$ is piecewise continuous periodic.

5.6b Inhomogenous w/Nonautonomous Homogeneous Part: $x' = A(t)x + g(t)$

For this even more general inhomogeneous equation, there is a variation of constant formula: $x(t_0) = x_0,\quad \Rightarrow\quad x(t) = \Phi_{t,t_0}x_0 + \int_{t_0}^t \Phi_{t,s}g(s)ds$.

5.7 Floquet Theory (FT) - Non-Autonomous DEQ

WIKI: Relates to the class of solutions to periodic linear differential equations of the form $\dot{x} = A(t)x$, with $A(t)$ a piecewise continuous periodic function with period T and through FT, the stability of solutions is revealed.

The main theorem of Floquet theory, Floquet's theorem, gives a canonical form for each fundamental matrix solution. FT gives a coordinate change $y = Q^{-1}(t)x$ with $Q(t + 2T) = Q(t)$ that transforms the periodic system to a traditional linear system with constant, real coefficients, $y' = Ry$.

Note that the solutions of the linear differential equation form a vector solution space. A matrix $\varphi(t)$ is called a **fundamental matrix solution** if all columns are linearly independent solutions. A matrix $\Phi(t)$ is called a **principal fundamental matrix solution** if all columns are linearly independent solutions and there exists t_0 such that $\Phi(t_0)$ is the identity. A principal fundamental matrix can be constructed from a fundamental matrix using $\Phi(t) = \varphi(t)\varphi^{-1}(t_0)$. The solution of the linear differential equation with the initial condition $x(0) = x_0$ is $x(t) = \varphi(t)\varphi^{-1}(0)x_0$ where $\varphi(t)$ is any fundamental matrix solution.

Floquet's Theorem

Let $\dot{x} = A(t)x$ be a linear first order differential equation, where $x(t)$ is a column vector of length n and $A(t)$ an $n \times n$ periodic matrix with period T (that is $A(t+T) = A(t)$ for all real values of t). Let $\varphi(t)$ be a fundamental matrix solution of this differential equation. Then, for all $t \in \mathbb{R}$, $\varphi(t+T) = \varphi(t)\varphi^{-1}(0)\varphi(T)$. Here $\varphi^{-1}(0)\varphi(T)$ is known as the **monodromy matrix**. In addition, for each matrix B (possibly complex) such that $e^{TB} = \varphi^{-1}(0)\varphi(T)$, there is a periodic (period T) matrix function $t \to Q(t)$ such that $\varphi(t) = Q(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R. and a real periodic (period $2T$) matrix function $t \to Q(t)$ such that $\varphi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Consequences and applications

This mapping $\varphi(t) = Q(t)e^{tR}$ gives rise to a time-dependent change of coordinates ($y = Q^{-1}(t)x$), under which our original system becomes a linear system with real constant coefficients $\dot{y} = Ry$. Since $Q(t)$ is continuous and periodic it must be bounded. Thus the stability of the zero solution for $y(t)$ and $x(t)$ is determined by the eigenvalues of R.

The representation $\varphi(t) = Q(t)e^{tB}$ is called a Floquet normal form for the fundamental matrix $\varphi(t)$. The eigenvalues λ_i of e^{TB} are called the **characteristic multipliers** of the system. They are also the eigenvalues of the (linear) Poincaré maps $x(t) \to x(t+T)$. A **Floquet exponent** (sometimes called a **characteristic exponent**), is a complex μ such that $e^{\mu T}$ is a characteristic multiplier of the system. Notice that Floquet exponents are not unique, since $e^{\left(\mu + \frac{2\pi i k}{T}\right)T} = e^{\mu T} =: \lambda_i$, where k is an integer. The real parts of the Floquet exponents are called Lyapunov exponents. The zero solution is asymptotically stable if all **Lyapunov exponents** are negative, Lyapunov stable if the Lyapunov exponents are nonpositive and unstable otherwise.

Another Perspective:

So with Poincaré Maps, we have a theory which dealt with one-dimensional periodic orbits, this theory helps us with periodic orbits of higher dimension. So, say we are given a system $\dot{x} = \vec{f}(x)$ with $\vec{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^n$. And assume that the system has a periodic orbit of period T where $\Gamma : x = \gamma(t), \quad 0 \le t \le T$, contained in E. The derivative of the Poincaré map $D\vec{P}(x_0)$, at a point $x_0 \in \Gamma$ is an $(n-1) \times (n-1)$ matrix and one can show that if $\left|D\vec{P}(x_0)\right| < 1$, then the periodic orbit Γ is asymptotically stable.

The linearization of the system about Γ is defined as the non-autonomous linear system $\dot{x} = A(t)x, \quad [1]$

where $A(t) = D\vec{f}(\gamma(t))$ is a continuous T-periodic function.

The matrix $D\vec{P}(x_0)$ is determinable by a fundamental matrix for the linearization of the system about the periodic orbit Γ. If $\varphi(t)$ is a fundamental matrix for [1] which satisfies $\varphi(0) = I$, then $\left|D\vec{P}(x_0)\right| = |\varphi(T)|$ for any point $x_0 \in \Gamma$. It then follows from the above theorem that $\left|D\vec{P}(x_0)\right| = |e^{TB}|$. The eigenvalues of e^{BT} given by $e^{\lambda_j T}$ where λ_j, $j = 1, 2, \ldots, n$, are the eigenvalues of the matrix B. The eigenvalues λ_j of B are called **characteristic exponents** of $\gamma(t)$, and the eigenvalues $e^{\lambda_j T}$ of e^{BT} are called the **characteristic multipliers** of $\gamma(t)$.

From the notes: $\dot{x} = A(t)x$

An interesting special case of nonautonomous equations are time periodic equations, $A(t) = A(t+T)$ for all $t \in \mathbb{R}$ and some fixed. $T > 0$. In this case, $\Phi_{t+T,s+T} = \Phi_{t,s}$, since this is true at $t = s$ and both sides, considered as functions in t, solve the same ODE $Y' = A(t)Y$, then $\partial_t \Phi_{t+T,s+T} = A(t+T)\Phi_{t+T,s+T} = A(t)\Phi_{t+T,s+T}$, and $\partial_t \Phi_{t,s} = A(t)\Phi_{t,s}$. As a consequence, the fundamental matrix factors $\Phi_{t,0} = \Phi_{t-kT,0}\Phi_{T,0}^k$, and is determined for all times by $\Phi_{t,0}$, with $t \in [0, T]$, only.

Theorem 8 (**Floquet theory**, $X = \mathbb{C}^n$). There is $B \in \mathbb{C}^{n \times n}$, $Q(t) = Q(t+T) \in \mathbb{C}^{n \times n}$ invertible with $Q(0) = id$, such that $\Phi_{t,0} = Q(t)e^{Bt}$.

Remark 5.7 (**Floquet as coordinate change to normal form**). We can think of $Q(t)$ as a time-dependent coordinate change that makes the equation autonomous. Indeed, $e^{B(t-s)} = Q(s)^{-1}\Phi_{t,s}Q(t)$ (which is the form of a solution for an autonomous equation $x'(t-s) = Bx$). In a commutative diagram for the evolution of the vector bundle ($x \in \mathbb{C}^n, t \in \mathbb{R}$):

$$\begin{array}{ccc} (\mathbb{C}^n, s) & \xrightarrow{e^{B(t-s)}} & (\mathbb{C}^n, t) \\ \downarrow Q(s) & & \downarrow Q(t) \\ (\mathbb{C}^n, s) & \xrightarrow{\Phi_{t,s}} & (\mathbb{C}^n, t) \end{array}$$

Corollary 5.8 (**Spectral Mapping**): Floquet exponents $\{\alpha_j\} = \log\sigma(\Phi_{T,0})$ are logarithms of eigenvalues of the period map: $e^{\sigma(B)T} = \sigma(\Phi_{T,0})$, preserving algebraic and geometric multiplicities and lengths of individual Jordan chains. As a consequence, all solutions to linear time periodic equations can be written as complex functions: $x(t) = \Sigma_\lambda p_\lambda(t)e^{\alpha t}$, where the vector valued functions p_λ are polynomials in t with order at most $alg(\lambda) - 1$, $\lambda = e^{\alpha T}$, and T-periodic coefficients.

Lemma 5.9 (**real Floquet theory**): Assume that $A(t)$ is real. There exists a real B (and $Q(t)$) in theorem 8 if and only if all Jordan blocks to eigenvalues $\lambda_j < 0$ come in pairs, that is, there is an even number of blocks of any given length d.

Corollary 5.10: Assume $A(t)$ is real and T-periodic. Then there exists a real $2T$-periodic Floquet transformation $Q(t) = Q(t+2T)$, and $e^{2BT} = \Phi_{2T,0}$.

We note that floquet exponents are not unique, since the logarithm is only determined up to a shift by $2\pi i$. One therefore sometimes fixes floquet exponents in $|Im(\alpha)| \leq 2\pi/T$.

8 Center Manifolds, Normal Forms, and Bifurcations

Bistability: When a system has two stable equilibrium states. Something that is bistable can be resting in either of two states. These rest states need not be symmetric with respect to stored energy. An example of a mechanical device which is bistable is a light switch. The switch lever is designed to rest in the "on" or "off" position, but not between the two. Bistable behavior can occur in mechanical linkages, electronic circuits, nonlinear optical systems, chemical reactions, and physiological and biological systems.

In a conservative force field, bistability stems from the fact that the potential energy has two local minima, which are the stable equilibrium points. By mathematical arguments, a local maximum, an unstable equilibrium point, must lie between the two minima. At rest, a particle will be in one of the minimum equilibrium positions, because that corresponds to the state of lowest energy. The maximum can be visualized as a barrier between them.

We model a system with bistability and possibility of switching by a simple scalar ODE with cubic nonlinearity

$$u' = f(u;a); \qquad f(u;a) = u(1-u)\left(u - \tfrac{1}{2}\right) + a \qquad (8.1)$$

This DEQ has three f.p.s: $\{u_s, u_m, u_\ell\} = \left\{0, \tfrac{1}{2}, 1\right\}$.

Coupling: Interesting dynamics can arise when such systems are coupled on a lattice or a graph. The simplest example is: $u_1' = d(u_2 - u_1) + u_1(1-u_1)\left(u_1 - \tfrac{1}{2}\right) + a,$

$$u_2' = d(u_1 - u_2) + u_2(1-u_2)\left(u_2 - \tfrac{1}{2}\right) + a. \qquad (8.2)$$

Weak coupling (around $d = 0$). Setting $d = 0$, we readily find that there are 9 equilibria in the bistable regime, $(u_1, u_2) = (u^i, u^j)$, where $i,j \in \{u_s, u_m, u_\ell\}$. Linearizing the equation (8.2) with $u_j' = 0$ at these solutions, we find a diagonal matrix with entries $f'(u_j) = \begin{bmatrix} -\left(d + \frac{1}{2}\right) & d \\ d & -\left(d + \frac{1}{2}\right) \end{bmatrix}$, that are non-zero in the bistable regime. The implicit function theorem therefore guarantees that there are precisely 9 solutions branching from these f.p.s for $|d| < 1$, sufficiently small, for any a in the bistable regime. One can also compute stability in the limit $d = 0$, finding that the number of unstable eigenvalues equals the number of components which take the value u_m (middle equilibrium).

Strong coupling ($d >> 1$). With strong coupling, one may expect (and can prove) that all equilibria are of the form $u_1 = u_2$. To see this, set $d = \frac{1}{\varepsilon}$, multiply the right-hand side of (8.2) by ε, add, and subtract, to find:

$$\text{add: } \varepsilon\left(u_1(1-u_1)\left(u_1 - \tfrac{1}{2}\right) + u_2(1-u_2)\left(u_2 - \tfrac{1}{2}\right) + 2a\right) = \varepsilon(f(u_1) + f(u_2)) = 0,$$

$$\text{sub: } 2(u_2 - u_1) + \varepsilon\left(u_1(1-u_1)\left(u_1 - \tfrac{1}{2}\right) - u_2(1-u_2)\left(u_2 - \tfrac{1}{2}\right)\right) = 2(u_2 - u_1) + \varepsilon(f(u_1) - f(u_2)) = 0.$$

When $\varepsilon \neq 0,$ first equation can be divided by ε. As $\varepsilon \to 0$ one readily finds from the second equation that, $u_1 = u_2$, which implies using the first equation that $f(u_2) = f(u_1) = 0$. So, we have an equilibrium. If $f'(u_1) \neq 0$, We can solve the equation for $\varepsilon << 1$ with the implicit function theorem, and find unique solutions near $\varepsilon = 0$, since the linearization at equilibria is $\partial_u F = \begin{bmatrix} f'(u_1) & f'(u_1) \\ 2 & -2 \end{bmatrix}$, so $\det \partial_u F \neq 0$. Since $u_1 = u_2,$ $f(u_1) = 0$ solves the equation for $\varepsilon \neq 0$ as well, we conclude that (locally) all solutions have $u_1 = u_2$. One can easily extend this argument to show that also globally there are no solutions with $u_1 \neq u_2$, provided $|\varepsilon| << 1$.

Intermediate d, **critical** a. Clearly, the most interesting question here is locating when the number of equilibria and/or their stability changes. Increasing a from 0 past the boundary of bistability, we expect this to occur multiple times at least for small values of d, since the number of equilibria changes from 9 to 3 for small d.

One could also ask for minimal values of d such that all equilibria are of the form $u_1 = u_2$. More generally, one would like to describe equilibria and their stability in larger coupled systems, with $u_j' = d((Au)_j - u_j) + f(u_j a),$ where the $n \times n$-matrix A represents a weighted graph on labels $1 \leq j \leq n$ and $\Sigma_\ell A_{k\ell} = 1$.

8.2 Continuation

The implicit function theorem guarantees that "typically," the solution set of a smooth DEQ does not change qualitatively when the parameter is changed slightly.

Theorem 15 (**IFT**). Suppose $F \in C^k,\ k = 1, \ldots, \infty, \omega$, $F(u_*; \mu_*) = 0$, and $\partial_u F(u_*; \mu_*)$ is invertible. Then there exists locally unique $\varphi(\mu)$ such that $F(\varphi(\mu); \mu) = 0$.

A naive approach to finding equilibria would then be the following basic continuation algorithm:

(i) Make an initial guess $(\overline{u}, \overline{\mu})$;

(ii) Converge Newton's method to find solution $(u_*, \overline{\mu})$; initialize $u_{+old} := u_*,\ \mu_{old} = \overline{\mu}$;

(iii) Step in the parameter $\mu_{new} := \mu_{old} + \delta$;

(iv) Converge Newton method with (naive) initial guess u_{old} at μ_{new} to find u_{new};

(v) Reinitialize $u_{old} := u_{new},\ \mu_{old} := \mu_{new}$, and go to step (iii).

The implicit function theorem, when it applies, guarantees that this algorithm will perform well, locally. We can now attempt to continue globally - until problems arise.

Clearly, we expect this algorithm to fail when the implicit function theorem fails. If we think of $det \partial_u F(\varphi(\mu; \mu))$ as a function along the branch, we expect nonzero values for almost all values of μ, but we also expect to encounter unavoidable sign changes, at which point our algorithm breaks down. A remedy is the following idea.

8.3 Arclength Continuation

We think of $F(u; \mu)$ as a function of the $n + 1$ variables $v = (u, \mu)$ and hope that the derivative with respect to both is onto the tangent space. The following theorem concludes that in this case the solution set of $F = 0$ is a smooth curve, locally, albeit not always given as a graph $u = \varphi(\mu)$.

Full Rank Theorem (which means the image has full dimensions). Suppose $F \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, $F(u_*) = 0$, and $DF(u_*)$ is onto $(\text{Img}(DF(u_*)\vec{v}) = \mathbb{R}^n$ for $\vec{v} \in \mathbb{R}^m$, in particular $m > n)$. Then the set of zeros of F near u_* is a C^k-manifold of dimension $m - n$.

Sards Theorem. Suppose $F \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, with $k > 1$ if $m \leq n$ and $k > m - n + 1$, otherwise. Let C be the critical points, that is, $DF(u)$ does not have maximal rank for $u \in C$. Then $F(C)$, the set of critical values, has Lebesgue measure zero.

Corollary 8.1 (**Generic smooth bifurcation curves**). Any smooth one-parameter family of vector fields $F(u; \mu)$ can be approximated by vector fields $F(u; \mu) + \varepsilon_k$, with $\varepsilon_k \to 0$, such that the equilibria $F(u; \mu) + \varepsilon_k = 0$ form smooth curves in $\mathbb{R}^n \times \mathbb{R}$ for each ε_k.

In practice, the arclength continuation algorithm proceeds as follows:

Use the system: $F(v) = 0, \quad \langle v - v_*, e \rangle = 0 \qquad (8.3)$

(i) Initialize continuation by finding solution $v_* = (u_*, \mu_*)$; initialize w/a chosen ds arclength step size;

(ii) Start loop;

(iii) Find kernel e of $\partial_v F(v_*)$; normalize $|e| = 1$;

(iv) Step $v_* := v_* + ds \cdot e$;

(v) Solve (8.3) for v using Newton's method with i.c. v_* ;

(vi) Set $v_* := v$;

(vii) End loop;

Finding the kernel is usually accomplished by solving the following (e in $\ker DF(v_*)$ and not zero):

$$DF(v_*)e = 0; \qquad (e, e_{old}) = 1, \qquad (8.4)$$

where e_{old} is in the kernel from the previous step, which we initialize in the very first instance as $(0,\dots,0,\pm 1)$, where the sign determines the direction of continuation. The system (8.4) is a linear equation in e which can be readily solved using the linear algebra available. One can also verify that this linear equation is simply $L \cdot e = r$, where $r = (0,\dots,0,1)^T$ and L is the derivative of (8.3) with respect to v (which one may wish to provide anyway for Newton's method). Alternatively, one can find an approximate kernel in (iii) using the **secant continuation method**: $e := v_* - v, \quad (8.5)$

where v is the solution from the previous continuation step. One needs to initialize in this case with two solutions, finding for instance solutions $u_1; u_2$ for parameter values (μ_1, μ_2).

Lyapunov–Schmidt Reduction (L-S)

This is used to study solutions to nonlinear equations in the case when the implicit function theorem does not work. It permits the reduction of infinite-dimensional equations in Banach spaces to finite-dimensional equations. Locally solving equations $F(u;\mu) = 0$ when the implicit function theorem is not available. It is usually accomplished in two steps. The first step we describe in this section as a reduction step.

Theorem 18 (**Lyapunov-Schmidt reduction**). Suppose $F = F(u;\mu) \in C^k(\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$, $k = 1,\dots,\infty,\omega$, $F(0;0) = 0$. Let P and Q be projections onto kernel and range of $\partial_u F(0;0)$. Then there exists a locally defined C^k-function $\varphi : RgP \times \mathbb{R}^p \to RgQ$ (or $\varphi(\ker;\mu) \to RgQ$), such that solutions $(u;\mu)$ to $F(u;\mu) = 0$ are in one-to-one correspondence to solutions to $\varphi(\widetilde{u};\mu) = 0$.

Problem Set up

Let $F(x,\lambda) = 0$ be the given nonlinear equation, and X, Λ, Y be Banach spaces (Λ is a parameter space). $F(x,\lambda)$ is C^p-map from a neighborhd of some point $(x_0,\lambda_0) \in X \times \Lambda$ to Y where the equation $F_0 := F(x_0,\lambda_0) = 0$ is satisfied. For the case the when linear operator $\partial_x F(x,\lambda)$ is invertible, IFT assures that there exists a solution $x(\lambda)$ satisfying the equation $F(x(\lambda),\lambda) = 0$ at least locally close to λ_0, so use continuation. In the opposite case, when the linear operator $\partial_x F(x,\lambda)$ is non-invertible, the Lyapunov–Schmidt reduction can be applied in the following way.

Assumptions: One assumes that the operator $\partial_x F(x,\lambda)$ is a Fredholm operator, that is: $\ker \partial_x F_0 =: X_1$ where X_1 has finite dimension, and the $range(\partial_x F_0) =: Y_1$ has finite co-dimension ($\dim Y_2 < \infty$) and is a closed subspace in Y. Without loss of generality, one can also assume that $(x_0,\lambda_0) = (0,0)$.

Lyapunov–Schmidt construction

Let us split Y into the direct product $Y = Y_1 \oplus Y_2 = range(\partial_x F_0) \oplus range(\partial_x F_0)^\perp$, where $\dim Y_2 < \infty$. Let Q be the projection operator onto Y_1. Consider also the direct product $X = X_1 \oplus X_2 = \ker(\partial_x F_0) \oplus \ker((\partial_x F_0)^\perp)$. Applying the operators Q and $I - Q$ to the original equation, one obtains the equivalent system: $QF(x,\lambda) = 0$, $(I-Q)F(x,\lambda) = 0$. Let $x = x_1 + x_2$ where $x_1 \in X_1$ and $x_2 \in X_2$, then the first equation: $QF(x_1 + x_2, \lambda) = 0$, can be solved with respect to x_2 by applying the **implicit function theorem** to the operator: $QF(x_1 + x_2, \lambda) : X_2 \times (X_1 \times \Lambda) \to Y_1$ (since $\partial_{x_2} QF_0$ is invertible, the conditions of the implicit function theorem are now fulfilled). Thus, there exists a unique solution $x_2(x_1,\lambda)$ satisfying: $QF(x_1 + x_2(x_1,\lambda),\lambda) = 0$. ("Stuff the kernel into the parameter space.")

However, the implicit function theorem doesn't tell you how to find x_2. If the correct projections are chosen, some qualitative properties of x_2 (such as continuity and differentiability) can be obtained such that a Taylor expansion can be used to approximate x. Now substituting $x_2(x_1,\lambda)$ into $(I-Q)F(x,\lambda) = 0$, one obtains the final finite-dimensional equation: $G(x_1) := (I-Q)F(x_1 + x_2(x_1,\lambda),\lambda) = 0$, for use in the following steps:

Indeed, this last equation is now finite-dimensional, since the range of $(I-Q)$ is finite-dimensional (projection onto $(range)^\perp$ of Q, which is finite since the codomain of $Y_1 = range(\partial_x F_0)$ is finite (by assumption). This equation is now to be solved with respect to x_1, which is finite-dimensional (by assumption), and parameters : λ.

From a practical computational perspective, one uses orthogonal projections:

$$Pu = \langle e, u\rangle e, \qquad Qu = (I - P)u = \langle e^*, u\rangle e^*, \qquad (8.9)$$

where $|e| = |e^*| = 1$ span the kernels of $\partial_u F$ and $(\partial_u F)^T$, respectively.

Computations sometimes simplify using **spectral projections**, which are suitable when $\langle e, e^*\rangle \neq 0$ (where $e \in \ker(\partial_u F)$ and $e^* \in \ker(\partial_u F)^T$), that is, e does not belong to the range of $\partial_u F$ (because of the Fredholm alternative: $\ker(\partial_u F)^T \perp Range(\partial_u F)$). One then defines $Pu = (1 - Q)u = \langle e^*, u\rangle e,$ with $\langle e, e^*\rangle = 1$.

Classification of bifurcations

One can view bifurcations as a failure of structural stability within a family. A starting point for classifying bifurcation types is the **Kupka-Smale theorem** that lists three generic properties of vector fields:

- Hyperbolic equilibrium points
- Hyperbolic periodic orbits
- Transversal intersections of stable and unstable manifolds of equilibrium points and periodic orbits.

Different ways that these Kupka-Smale conditions fail lead to different bifurcation types. Bifurcation theory constructs a layered graph of bifurcation types. These layers can be organized by the codimension of the bifurcation types. Codimension-n means you must vary n parameters for the bifurcation to occur. Also, there is a zero eigenvalue with algebraic multiplicity n.

Codimension one bifurcations comprise the top level of bifurcation types. Single failures of the Kupka-Smale properties yield the following types of **codimension one bifurcations**:

- **Equilibria**
  - Saddle-Node Bifurcation
  - Andronov-Hopf Bifurcation
- **Periodic Orbits**
  - Fold Limit Cycle Bifurcation
  - Flip Bifurcation (aka Period Doubling bifurcation)
  - Neimark-Sacker Bifurcation (aka Torus bifurcation)
- **Global Bifurcations**
  - Homoclinic Bifurcation of equilibria
  - Homoclinic tangencies of stable and unstable manifolds of periodic orbits
  - Heteroclinic Bifurcation of equilibria and periodic orbits

This is not a comprehensive list of codimension one bifurcations. Moreover, there are subcases in the list above that deal with such issues as whether an Andronov-Hopf bifurcation is sub-critical or super-critical.

The classification of bifurcation types becomes more complex as their codimension increases. There are five types of "local" **codimension two bifurcations** of equilibria:

Bautin Bogdanov-Takens Cusp Fold-Hopf Hopf-Hopf

8.5 Elementary Bifurcations –Saddle-Node (fold) and Transcritical Bifurcations

We assume that the kernel of $\partial_u F(0;0)$ is one-dimensional. Lyapunov-Schmidt reduction then yields a scalar bifurcation equation $\varphi(u_0;\mu) = 0$. One usually introduces coordinates $u_0 = Le$, $L \in \mathbb{R}$, and uses the multivariate Taylor expansion in (L,μ), $\varphi(Le;\mu) = \sum_{j,k=0}^{\infty} \varphi_{jk} L^j \mu^k.$ (8.11)

From the Lyapunov-Schmidt reduction, we find that $\varphi_{10} = 0$, that is, the linear term in L vanishes (otherwise, we could simply solve the reduced equation w/IFT for L).

Theorem 19 (**solution near saddle-node bifurcation**). Suppose we have a solution $F(0;0) = 0$, such that:

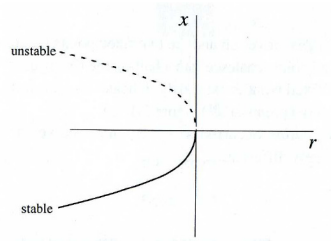
(i) the kernel of $\partial_u F(0;0)$ is one-dimensional, spanned by e, and

(ii) the reduced coefficients defined in (8.11) satisfy $\varphi_{01} \cdot \varphi_{20} \neq 0$ (transversality & non-degeneracy).

Let $\iota = -sign(\varphi_{01} \cdot \varphi_{20})$. Then solutions to $F(u;\mu) = 0$ exist near the origin if and only if $\iota\mu > 0$, in which case there is precisely one solution when $\mu = 0$ and precisely two solutions when $\mu \neq 0$, with expansion:
$u_{\pm}(\mu) = \pm(-\frac{\varphi_{01}\mu}{\varphi_{20}})^{\frac{1}{2}}\vec{e} + O(\mu)$. (8.12)

Reduced Coefficients: One finds, using orthogonal projections $Pu = \langle \vec{e}, u \rangle \vec{e}$ and $(1-P)u = \langle \vec{e}^*, u \rangle \vec{e}^*$ (where unit vectors $\vec{e}, \vec{e}^*$ span the kerns of $\partial_u F$ & $(\partial_u F)^T$), that $\varphi_{01} = \langle \partial_\mu F(0;0), \vec{e}^* \rangle$, and $\varphi_{20} = \langle \partial_{uu} F(0;0)\langle \vec{e}, \vec{e} \rangle, \vec{e}^* \rangle$.

From the Web:



SN is a local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other. The term "saddle-node bifurcation" is most often used in reference to continuous dynamical systems. In discrete dynamical systems, the same bifurcation is often instead called a fold bifurcation. Another name is blue skies bifurcation in reference to the sudden creation of two fixed points. If the phase space is one-dimensional, one of the equilibrium points is unstable (**the saddle**), while the other is stable (**the node**). Saddle-node bifurcations may be associated with **hysteresis loops**.

Normal form: A typical example of a differential equation with a saddle-node bifurcation is: $\frac{dx}{dt} = r + x^2$. Here x is the state variable and **r** is the bifurcation parameter.

- If $r < 0$ there are two equilibrium points, a stable equilibrium point at $-\sqrt{-r}$ and an unstable one at $+\sqrt{-r}$.
- At $r = 0$ (the bifurcation point) there is exactly one equilibrium point. At this point the fixed point is no longer hyperbolic. In this case the fixed point is called a **saddle-node fixed point**.
- If $r > 0$ there are no equilibrium points.

Saddle node bifurcation: A scalar differential equation $\frac{dx}{dt} = f(r,x)$ which has a fixed point at $x = 0$ for $r = 0$ with $\frac{\partial f}{\partial x}(0,0) = 0,$ is locally topologically equivalent to $\frac{dx}{dt} = r \pm x^2$, provided it satisfies $\frac{df}{dr}(0,0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$. The first condition is the **transversality condition** and the second condition is the **nondegeneracy condition**.

Hysteresis is the dependence of the state of a system on its history. For a system with hysteresis, a **hysteresis loop** is the loop formed by the function when the independent variable is increased and decreased repeatedly over the part of the domain responsible for the hysteresis.

Transcritical bifurcation (for example, the generalized logistic equation)

Sometimes a trivial solution $u \equiv 0$ is unchanged for all values of the parameter. Examples arise for instance in chemistry or ecology where absence of all species $u = 0$ is a trivial state. In this case, one is interested in the bifurcation of solution branches from this trivial state. Of course, IFT guarantees that $u = 0$ is the only solution for parameter values in a vicinity of 0 if $\partial_u F(0;0)$ is invertible.

Theorem 20 (**Transcritical bifurcation**). Suppose we have a trivial solution $F(0;r) = 0$, such that

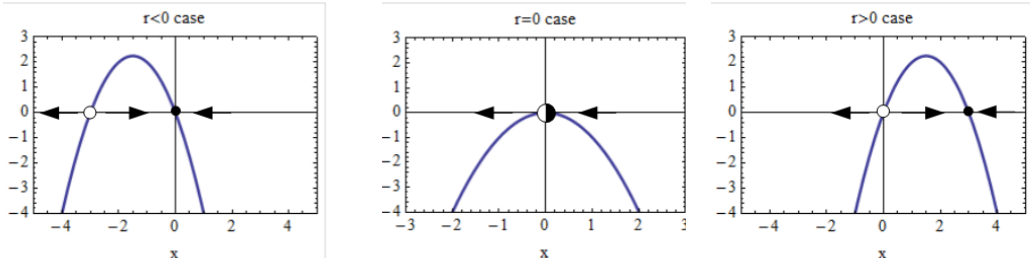
(i) the kernel of $\partial_u F(0;0)$ is one-dimensional, spanned by e, and

(ii) the reduced coefficients defined in (8.11) satisfy $\varphi_{11} \cdot \varphi_{20} \neq 0$.

Then there exists a unique solution $u \neq 0$ to $F(u;r) = 0$ for $r \neq 0$ sufficiently small with expansion:

$u_{\pm}(r) = -\frac{\varphi_{11} r}{\varphi_{20}} e + O(r^2)$. (8.13)

From the Web:



Inverted Logistic Standard Logistic

A fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied. In other words, both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide. So the unstable fixed point becomes stable and vice versa. (see images above)

Normal form: $\frac{dx}{dt} = rx - x^2$. This equation is similar to the logistic equation but in this case we allow r and x to be positive or negative. The two fixed points are at $x = 0$ and $x = r$. When the parameter r is negative, the fixed point at $x = 0$ is stable and the fixed point $x = r$ is unstable. But for $r > 0$, the point at $x = 0$ is unstable and the point at $x = r$ is stable. So the bifurcation occurs at $r = 0$.

A typical example (in real life) could be the **consumer-producer problem** where the consumption is proportional to the (quantity of) resource. For example: $\frac{dx}{dt} = rx(1-x) - px,$ where $rx(1-x)$ is the logistic equation of resource growth; and px is the consumption, proportional to the resource x.

8.6 Symmetries

If $\Gamma \subset O(n)$, a subset of the orthogonal group in n dimensions, we say that a vector field f is equivariant with respect to Γ if $f(\gamma u;\mu) = \gamma f(u;\mu),$ for all $\gamma \in \Gamma$. Setting $L := \partial_u f(0;0)$ and differentiating with respect to u at $u = 0$ shows that for all $\gamma \in \Gamma$, $L\gamma = \gamma L$. In particular, $Ker(L)$ is invariant under Γ (if $t \in Ker(L)$, $Lt = 0,$ so $\gamma Lt = 0$). In other words, Γ acts on $Ker(L)$. In a completely analogous fashion, L^T commutes with the action of Γ and the cokernel is invariant under Γ.

Lemma 8.3 (**Equivariant L-S**). If F is equivariant with respect to γ and we choose projections P & Q in the L-S Thm as orthogonal projections, then the Lyapunov-Schmidt reduced $\varphi(u_0;\mu) = 0$ is equivariant wrt the action of Γ on $Ker(L)$ and $coker(L)$.

We can summarize this discussion, concluding that reduced bifurcation problems are equivariant with respect to the representation of Γ on the kernel of $\partial_u F(0;0)$.

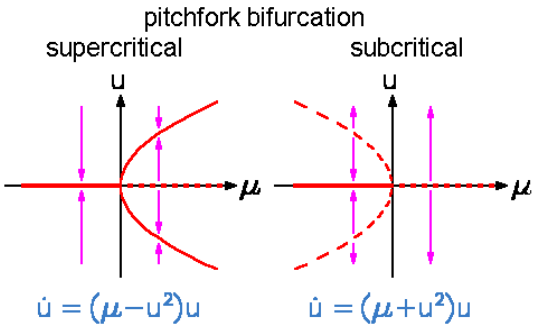
Pitchfork Bifurcation (PB)

One-dimensional bifurcations with symmetry — pitchforks. If the kernel of $L := \partial_u F(0;0)$ is one dimensional, the action of Γ on $Ker(L)$ is a subgroup of $O(1) = \mathbb{Z}_2$, hence either trivial or $\mathbb{Z}_2$. Consider for instance the action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ through $(u_1,u_2) = (u_2,u_1)$.

Theorem 21 (**Pitchfork bifurcation**). Suppose $F \in C^3(\mathbb{R}^n \times \mathbb{R};\mathbb{R}^n)$ s.t. $F(\gamma u;\mu) = \gamma F(u;\mu)$, for some $\gamma \in O(n)$, $F(0;0) = 0$, and $Ker(\partial_u F(0;0)) = span(\vec{e})$ is one-dimensional. Assume that $\gamma\vec{e} = -\vec{e}$ and that the Lyapunov-Schmidt-reduced equation has $\varphi_{11} \cdot \varphi_{30} \neq 0$ and $\varphi_{10} = \varphi_{20} = \varphi_{01} = 0$. Then the set of solutions u,μ in a neighborhood of the origin includes $u \equiv 0,$ and a unique branch ($u = \pm\sqrt{\frac{-\mu\varphi_{11}}{\varphi_{30}}}$) given through
$\mu = \frac{-\varphi_{30}}{\varphi_{11}}A^2 + O(A^4),\ u = A \cdot \vec{e} + O(A^2)$.

Remark 8.4. Clearly, the expansion for the solution is well determined if $\varphi_{30} \neq 0$, in which case the sign of φ_{30} determines the direction of branching, that is, if nontrivial solutions exist for $\mu > 0$ or $\mu < 0$.

From the Web:



PB is a local bifurcation where the system transitions from one fixed point to three fixed points. Pitchfork bifurcations, like Hopf bifurcations have two types – supercritical and subcritical. In continuous dynamical systems described by ODEs—i.e. flows—pitchfork bifurcations occur generically in systems with symmetry.

Supercritical Normal Form: $\frac{du}{dt} = \mu u - u^3$. For negative values of μ, there is one stable equilibrium at $u = 0$. For $\mu > 0$ there is an unstable equilibrium at $u = 0$, and two stable equilibria at $u = \pm\sqrt{\mu}$.

Subcritical Normal Form: $\frac{du}{dt} = \mu u + u^3$. In this case, for $\mu < 0$ the equilibrium at $u = 0$ is stable, and there are two unstable equilibria at $u \pm \sqrt{-\mu}$. For $\mu > 0$ the equilibrium at $u = 0$ is unstable.

Formal definition: An ODE $\dot{u} = f(u,\mu)$, described by a one parameter function $f(u,\mu)$ with $\mu \in \mathbb{R}$ satisfying: $-f(u,\mu) = f(-u,\mu)$, (f is an odd function symmetric about μ), where: $\partial_u f(0,\mu_0) = 0$ (one branch), $\partial_u^2 f(0,\mu_0) = 0$ (2 branches), $\partial_u^3 f(0,\mu_0) \neq 0$ (not 3 branches), $\partial_\mu f(0,\mu_0) = 0$ ($u \equiv 0$, is an equilibrium), and $\partial_{\mu u} f(0,\mu_0) \neq 0$ (away from $u = 0$, there is nonzero slope near $u = 0$) has a pitchfork bifurcation at $(u,\mu) = (0,\mu_0)$. The form of the pitchfork is given by the sign of the third derivative: $\partial_u^3 f(0,\mu_0)$ $\begin{cases} < 0, \text{ supercritical,} \\ > 0, \text{ subcritical.} \end{cases}$ (the curvature stabilizes)

Note that subcritical and supercritical describe the stability of the outer lines of the pitchfork (dashed or solid, respectively) and are not dependent on which direction the pitchfork faces. For example, the negative of the first ODE above, $\dot{u} = u^3 - \mu u$, faces the same direction as the first picture but reverses the stability.

Newton Polygon

A method for taking an implicit multivariate equation $f(u;\mu) = 0$, and determining the branches of solutions near a given solution (u,μ). Consider for example the equation: $f(u;\mu) := \mu^4 + \mu u + u^3 + \mu u^2 = 0.$ (8.16)

At $\mu = 0$, we have a triple solution (counting with complex multiplicity) $u = 0$ at the origin (note that if our solution is not at the origin, we can change variables such that it is), and we ask how these three solutions unfold. Newton's strategy is to plot exponents in the Taylor expansion of f on the positive lattice (called a Newton's polygon); see Figure 8.5.

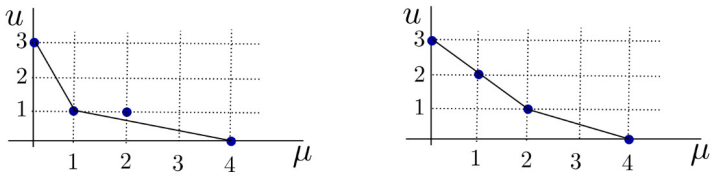


Figure 8.5: Exponents for (8.16) (left) and (8.17) (right) with associated Newton polygons as lower convex envelopes

One then draws the lower convex envelope. A line segment typically connect two exponents (which we call our **Leading Order Segment**), while all other exponents lie above/to the right of the span of this line segment. Although, sometimes more exponents can lie on the same line segment, while still most others lie above it, as seen in right panel of Fig 8.5 associated with (8.17 below). Each line segment gives a particular **scaling** that will make the terms associated with the powers on the line segment dominant terms, and terms above the line higher-order terms.

In the case of (8.16), we find two line segments. First, we look at the Leading Order Segment associated with the terms u^3, μu and therefore suggests that near $(0,0)$ we have:

- Scaling: $\mu \sim u^2$ which we can accomplish by:
- Setting: $u = u_1\varepsilon$ and $\mu = \varepsilon^2$. Which yields from (8.16): $\varepsilon^8 + \varepsilon^3(u_1 + u_1^3) + \varepsilon^4 u_1^2 = 0$.

Dividing by ε^3 and subsequently setting $\varepsilon = 0$, we find $u_1 + u_1^3 = 0$ and three solutions $u_1 = 0, \pm i$, which we can continue with the IFT (if those solutions were degenerate and IFT not applicable, we would simply apply this Newton polygon procedure to the scaled equation) to find for the non-trivial solutions $u = \pm i\mu^{1/2} + \frac{1}{2}\mu + O(|\mu|^{3/2})$.

So we have 2 of the 3 solutions, next the 2nd Order Segment of the Newton polygon gives us the expansion for the 3rd solution. We equate terms μu and μ^4, which near $(0,0)$ gives:

- Scaling: $u \sim \mu^3$ which we can accomplish by
- Setting: $u = u_2\varepsilon^3$ and $\mu = \varepsilon$. Which yields from (8.16): $\varepsilon^4 + \varepsilon^4 u_2 + \varepsilon^9 u_2^3 + \varepsilon^7 u_2^2 = 0$.

Dividing by ε^4 and subsequently setting $\varepsilon = 0$, we find $u_2 = -1 + \boldsymbol{O}(\mu^3)$, so $u = u_2\varepsilon^3 = -\mu^3 + O(\mu^4)$, and expanding further, $u = -\mu^3 - \mu^6$.

The case when several exponents lie on the same edge is illustrated by:

$f(u;\mu) := u^3 + u\mu^2 + 2\mu u^2 + \mu^4 = 0.$ (8.17) (see graph above)

Here, the Leading Order Segment near $(0,0)$ gives:

- Scaling: $\mu \sim u$ which we can accomplish by:
- Setting: $u = \varepsilon u_1$ and $\mu = \varepsilon$. Which yields from 8.17: $\varepsilon^3 u_1^3 + \varepsilon^3 u_1 + 2\varepsilon^3 u_1^2 + \varepsilon^4 = 0$.

Dividing by $\varepsilon^3 = 0$ and subsequently setting $\varepsilon = 0$, we find $u_1^3 + 2u_1^2 + u_1 = 0 \quad \Rightarrow \quad u_1 = 0$ or $u_1^2 + 2u_1 + 1 = 0 \quad \Rightarrow \quad u_1 = -1$ with multiplicity 2.

We therefore iterate our procedure. In order to get a **2nd order** correction v, we set $u_1 = -1 + v$ in 8.17b giving $(v-1)^3 + 2(v-1)^2 + (v-1) + \mu = v^3 - v^2 + \mu$, using the **second order** segment of the Newton polygon we have, $-v^2 \sim \mu \quad \Rightarrow \quad v = i\mu^{1/2}$. So, $u = u_1\varepsilon = \varepsilon(-1+v) = \varepsilon(-1 + i\mu^{1/2})$, and gathering all the calculated solutions: $u \in \left\{0,\ -\varepsilon,\ \varepsilon(-1 + i\mu^{1/2})\right\}$.

Inspecting the algorithm, one easily convinces oneself that it always yields the correct number of solution branches corresponding to the lowest pure power of u (if all terms possess a factor μ, one simply divides by this prefactor). A general theorem states that for analytic functions f, this procedure always terminates and finds all solutions in terms of power series in some root $\mu^{1/q}$.

8.9 Center Manifolds [Robinson pg 200]

Hyperbolic (**Stable/Unstable**) **Manifold Theorem**: Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^k map for $1 \le k \le \infty$ with $f(p) = p$ (equilibrium point). Then, $\mathbb{R}^n$ splits into the eigenspaces of Df_p, as $\mathbb{R}^n = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$, which correspond to the eigenvalues of Df_p greater than 1, equal to 1, and less than 1. There exists a neighborhood of p, $\boldsymbol{\mathcal{U}}_p \subset U$, such that $W^s(p,N_p,f)$ and $W^u(p,N_p,f)$ are C^k manifolds tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively, and are characterized by the exponential rate of convergence of orbits to p as follows. $W^{s/u}$ are positive/negative invariant. And given initial condition $x(t_0) \in W^{s/u}$, we have $\lim_{t\to\infty} x(t) = 0$, and $\lim_{t\to-\infty} x(t) = 0$, respectively. More specifically, if we assume that $0 < \mu < 1 < \lambda$ and norms on $\mathbb{E}^u$ and $\mathbb{E}^s$ are chosen such that $|Df_p|\mathbb{E}^s| < \mu$ and $m(Df_p|\mathbb{E}^u) > \lambda$. Then, $W^s(p,N_p,f) = \left\{q \in N_p : d\left(f^j(q),p\right) \le \mu^j d(q,p)\right\}$ for all $j \ge 0$ and $W^u(p,N_p,f) = \{q \in N_p : d(q_{-j},p)$

$\le \lambda^{-j} d(q,p)$ for all $j \ge 0$ where $\{q_{-j}\}_{j=0}^{\infty}$ is some choice of a past history of $q\}$.

Center Manifold Theorem: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^k map for $1 \le k \le \infty$ with $f(0) = 0$. Let k' be chosen to be (i) k if $k < \infty$ and (ii) some integer k' with $1 \le k' < \infty$ if $k = \infty$. Assume that $0 < \mu < 1 < \lambda$, and norms on $\mathbb{E}^u$ and $\mathbb{E}^s$ are chosen such that $|Df_p|\mathbb{E}^s| < \mu$ and $m(Df_p|\mathbb{E}^u) > \lambda$. Let $\varepsilon > 0$ be small enough so that $|Df_p|\mathbb{E}^s| < \mu - \varepsilon$ and $m(Df_p|\mathbb{E}^u) > \lambda + \varepsilon$. Let $r > 0$ and define an extension $\bar{f}: \mathbb{R}^n \to \mathbb{R}^n$ to be a C^k map with $\bar{f}|B(0,r) = f|B(0,r)$, $\left|\bar{f} - Df_0\right|_{C^1} < \varepsilon$, and $\bar{f} = Df_0$ off $B(0,2r)$. If $r > 0$ is small enough (it can depend on k'), then there exists an invariant $C^{k'}$ center-stable manifold, $W^{cs}(0,\bar{f})$, which is a graph over $\mathbb{E}^c \oplus \mathbb{E}^s$, which is tangent to $\mathbb{E}^c \oplus \mathbb{E}^s$ at 0, and which is characterized as follows: $W^{cs}(0,\bar{f}) = \left\{q : d\left(\bar{f}^j(q), 0\right)\lambda^{-j} \to 0 \text{ as } j \to \infty\right\}$.

This means that $\bar{f}^j(q)$ grows more slowly than λ^j. Similarly, there exists an invariant $C^{k'}$ center-unstable manifold, $W^{cu}(0,\bar{f})$, which is a graph over $\mathbb{E}^u \oplus \mathbb{E}^c$, which is tangent to $\mathbb{E}^u \oplus \mathbb{E}^c$ at 0, and which is characterized as follows: $W^{cu}(0,\bar{f}) = \{q : d(q_{-j}, 0)\mu^j \to 0 \text{ as } j \to \infty \text{ where } \{q_{-j}\}_{j=0}^{\infty} \text{ is some choice of a past history of } q\}$.

This means that q_{-j} grows more slowly than μ^{-j} as $j \to \infty$, or $-j \to -\infty$. Then, the center manifold of the extension $\bar{f}$ is defined as: $W^c(0,\bar{f}) = W^{cs}(0,\bar{f}) \cap W^{cu}(0,\bar{f})$. It is $C^{k'}$ and tangent to $\mathbb{E}^c$. There are also local center-stable, local center-unstable, and local center manifolds of f defined as:

$W^{cs}(0, B(0,r), f) = W^{cs}(0,\bar{f}) \cap B(0,r)$,

$W^{cu}(0, B(0,r), f) = W^{cu}(0,\bar{f}) \cap B(0,r)$, and

$W^{c}(0, B(0,r), f) = W^{c}(0,\bar{f}) \cap B(0,r)$, respectively.

These local manifolds depend on the extension $\bar{f}$; but if $f^j(q) \in B(0,r)$ for all $-\infty < j < \infty$, then $q \in W^c(0,\bar{f})$ for any extension $\bar{f}$ and $q \in W^c(0, B(0,r), f)$.

Center manifold reduction - statement of main result. Recall our previous discussion focused on solving $f(u;\mu) = 0$, $f \in C^k(\mathbb{R}^n \times \mathbb{R}^p; \mathbb{R}^n)$, using a reduction method roughly structured as follows:

Trivial solution $f(0;0) = 0$;

Linear approximation $\partial_u f(0;0)v = 0$, hence $v \in Ker(\partial_u f(0;0))$;

Reduce to $\varphi : Ker(\partial_u f) \times \mathbb{R}^p \to Coker(\partial_u f(0;0))$.

The results in this section reproduce this reduction for the equation: $u' - f(u;\mu) = 0$; where we are looking for solutions $u(t)$; $t \in \mathbb{R}$ that remain in a vicinity of the origin. The reduction here is in fact a reduction to a differential equation rather than an algebraic equation. Following roughly the previous outline, we aim for a result where:

Trivial solution $f(0;0) = 0$, $u(t) \equiv 0$ at $\mu = 0$;

Linear approximation $v' - \partial_u f(0;0)v = 0$, v "bounded" for all $t \in \mathbb{R}$, hence $v \in \mathbb{E}^c$;

Reduce to $u_c' = \varphi(u_c;\mu)$, $\varphi : \mathbb{E}^c \times \mathbb{R}^p \to \mathbb{E}^h$.

Here, $\mathbb{E}^c$ is the generalized eigenspace to $\partial_u f(0;0)$ corresponding to eigenvalues on the imaginary axis. In Jordan Normal Form, $\mathbb{E}^c$ is the span of all Jordan blocks corresponding to eigenvalues $\lambda \in i\mathbb{R}$.

Center Manifolds Theorem: Suppose $f \in C^k$, $k < \infty$. Then there exists a neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^n \times \mathbb{R}^p$ and a function: $h : \mathcal{U} \cap (\mathbb{E}^c \times \mathbb{R}^p) \to \mathbb{E}^h$, $h(0,0) = 0$, $h \in C^k$, such that $W^c = graph(h)$, the center manifold, is:

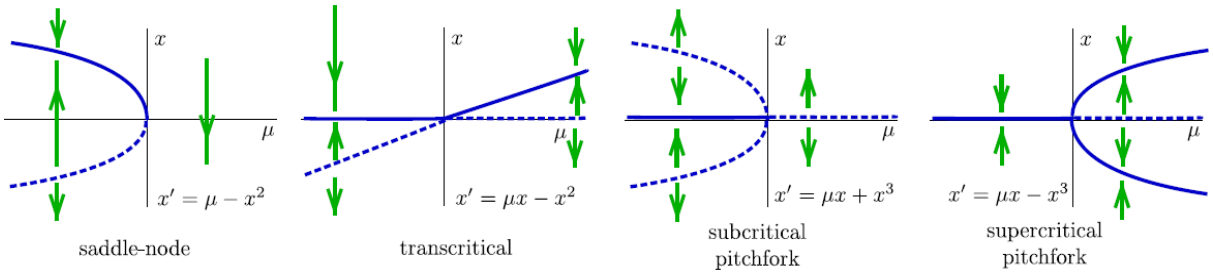
Locally invariant: If $u(0) \in W^c$, $u(t) \in \mathcal{U}$ for $t \in J$, $J = [t_-, t_+]$ containing 0, then $u(t) \in W^c$ for $t \in J$;

Maximal: $u(t) \in \mathcal{U}$ for $t \in \mathbb{R}$ implies $u(t) \in W^c$ for all t;

Tangent to $\mathbb{E}^c$: $\partial_u h(0,0) = 0$;

Stable: If $\mathbb{E}^u = \{0\}$: there exists $C, \eta > 0$ such that, if $u(t) \in \mathcal{U}$ for all $t \ge 0$, there exists $z(t) \in W^c$ for all $t \ge 0$ such that $|u(t) - z(t)| \le Ce^{-\eta t}$, for all $t \ge 0$.

Remark: The center manifold is not unique in general. Uniqueness can usually be guaranteed only if all solutions on the center manifold are bounded and contained in $\mathcal{U}$ for all $t \in \mathbb{R}$. The center manifold is also not C^∞, even when $f \in C^\omega$ is analytic.



Hopf Bifurcation: The term Hopf bifurcation refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes called dynamic (as opposed to static) bifurcation theory. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary as the parameter changes. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.

That a periodic solution should be generated in this event is intuitively clear from its graph. When the real parts of the eigenvalues are negative the fixed point is a stable focus; when they cross zero and become positive the fixed point becomes an unstable focus, with orbits spiralling out. But this change of stability is a **local change** and the phase portrait sufficiently far from the fixed point will be qualitatively **unaffected**: if the nonlinearity makes the far flow contracting then orbits will still be coming in and we expect a periodic orbit to appear where the near and far flow find a balance.

Another Perspective on Hopf

Next to the fold bifurcation, the second generic codimension-one bifurcation is the Hopf-bifurcation. We therefore consider a smooth vector field: $u' = f(u;\mu)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, with $f(0;0) = 0$, and $L = \partial_u f(0;0)$. Recall that algebraically simple eigenvalues of L are simple roots of the characteristic polynomial and therefore depend smoothly on parameters.

Normal Form: $u' = u(r - |u|^2)$

Hopf Bifurcation: Assume that L possesses a pair of algebraically simple eigenvalues $\pm i\omega_*$ with eigenvectors $e_*, \overline{e}_*$, and no other spectrum on the imaginary axis. Denote by $u_*(\mu)$ the unique equilibrium near the origin, $u(0) = 0$ and let $\lambda(\mu)$ be the smooth continuation of the eigenvalue $i\omega$ for the family of matrices $L(\mu) = \partial_u f(u(\mu);\mu)$. Assume that $Re\lambda'(0) \neq 0$. Then there exists a local family of periodic solutions $u_p(t;\tau)$, $0 \leq \tau \leq \tau_0$ with frequency $\omega(\tau)$, $\omega_p(0) = \omega_*$, for parameter values $\mu = \mu_p(\tau)$. Moreover, $u_p(t;\tau)$, period $T(\tau)$, and $\mu(\tau)$ are smooth functions of τ, and:

$u_p(t;\tau) = \tau e^{i\omega_p(\tau)t} e_* + c.c. + O(\tau^2)$, $\mu_p(\tau) = \mu_2\tau^2 + O(\tau^3)$, $\omega_p(\tau) = \omega_2\tau^2 + O(\tau^3)$, for some constants μ_2, ω_2.

One finds that periodic orbits are asymptotically stable when they bifurcate from an equilibrium which is unstable, and unstable in the opposite scenario.

8.11 Normal Form

A normal form is a simplified form that can be useful in determining the system's behavior. Normal forms are often used for determining local bifurcations in a system. All systems exhibiting a certain type of bifurcation are locally (around the equilibrium) topologically equivalent to the normal form of the bifurcation. For example, the normal form of a **saddle-node bifurcation** is: $x' = \mu + x^2$, where r is the bifurcation parameter. The **transcritical bifurcation** $x' = r\ln x + x - 1$ near $x = 1$ can be converted to the normal form $u' = \mu u - u^2 + O(u^3)$ (with the process described below) with the transformation $u = x - 1$, $\mu = r + 1$.

Normal Form Theory: (suppressing parameter dependence for this discussion)

Given: $x' = Lx + g(x) \in \mathbb{R}^n, \quad g \in C^\infty, \quad g(x) = O(|x|^2) \in \mathbb{R}^n$. Our goal is to simplify the nonlinearity g in a neighborhood of the origin, possibly removing it completely. We therefore try coordinate changes close to the identity: $x = y + \Phi(y)$, where $\Phi(y) = O\left(|y|^2\right)$. (e.g. $\Phi = \begin{bmatrix} y_1y_2 + y_2^2 + y_1y_2y_3 & y_1^2 + y_2y_3 + y_3^3 \end{bmatrix}$)

Let: $y := (y_1, \ldots, y_n)$, $\underline{j} =: (j_1, \ldots, j_n)$, $y^{\underline{j}} =: y_1^{j_1} \cdot \ldots \cdot y_n^{j_n}$, $\Phi(y) := \Phi_\ell \cdot y^{(\ell)} =: \sum_{|\underline{j}|=\ell} \Phi_{\underline{j}} y^{\underline{j}}$ with $\Phi_{\underline{j}} \in \mathbb{R}^k$ for some k, and $(ad_\ell L)(\Phi, y) := L(\Phi(y)) - D(\Phi(y))Ly$, where $L(\Phi(y))$ comes from the linear part, and $D(\Phi)Ly$ is the extra coming from the derivative of the LHS of our DEQ.

Normal Form Proposition 1: Let $H_\ell \leq Y_\ell, \quad 2 \leq \ell \leq k$, be a subspace of the homogeneous polynomials of degree ℓ such that: $Rg(ad_\ell L) \oplus H_\ell = Y_\ell$. Then there exists a sequence of polynomial transformations $id + \Phi_\ell \cdot y^{(\ell)}$ with $y^{(\ell)}$ homogeneous of degree $\ell = 2, \ldots, k$ such that the equation after the transformation at order k is in the form: $y' = Ly + g^{nf}(y) + O\left(|y|^{k+1}\right), \quad g^{nf}(y) = \sum_{\ell=2}^{k} g_\ell^{nf} \cdot y^{(\ell)}$. The resulting equation is then said to be in normal form.

We emphasize that we can perform this procedure to any order when the vector field is sufficiently smooth. However, even for analytic vector fields, the sequence of normal form transformations and the normal form itself, both obtained as formal power series, typically do not converge, such that one usually needs to consider a (very small) remainder of the vector field that is not in normal form.

Normal Form Proposition 2: We can choose $H_\ell = ker(ad_\ell(L^*))$, that is: $ker(ad_\ell(L^*) \oplus R_g(ad_\ell L)) = Y_\ell$.

Normal Form Corollary: A sequence of normal form transformations can achieve $ad_\ell(L^*) = (ad_\ell(L))^*$ up to any order: $g_{new}(e^{L^*\varphi}y) = e^{L^*\varphi} g_{new}(y)$, for all $\varphi \in \mathbb{R}^n$. In other words, we added a continuous symmetry to the nonlinear equation by choosing normal form transformations. Unfortunately, the linear part does in general not commute with this symmetry, whenever $LL^* \neq L^*L$.

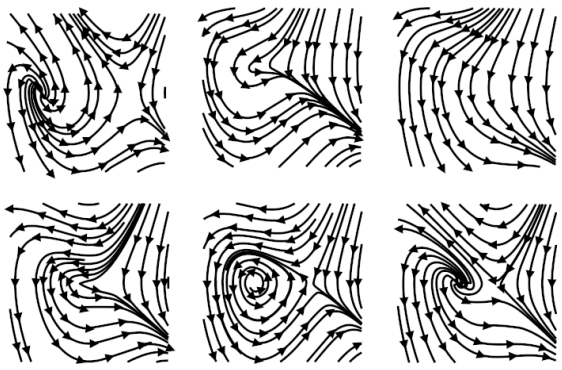
Remark (**Symmetries**). The transformations Φ can be chosen to preserve structure. If for instance $g(\gamma x) = \gamma g(x)$, for all $\gamma \in O(n)$, then we can choose Φ to commute with Γ and the normal form will commute with Γ. One therefore carries out all computations in the Lie algebra of γ-equivariant polynomials. Lastly, if g is a Hamiltonian vector field, we can choose $id + \Phi$ as canonical transformations and the normal form is Hamiltonian, as well. In all those cases, calculations are performed within the Lie algebra of vector fields with the prescribed structure.

Bogdanov–Takens Bifurcation:

Wiki: BT is well-studied example of a bifurcation with co-dimension two (meaning that two parameters must be varied for the bifurcation to occur). A system $y' = f(y)$ undergoes a Bogdanov–Takens bifurcation if it has a fixed point, and the linearization of f around that point has a double eigenvalue at zero (assuming some technical nondegeneracy conditions are satisfied).

Three codimension-one bifurcations occur nearby: a **saddle**-**node** bifurcation, an Andronov–**Hopf** bifurcation and a **homoclinic** bifurcation. All associated bifurcation curves meet at the Bogdanov–Takens bifurcation.

The normal form of the Bogdanov–Takens bifurcation is: $y_1' = y_2, \qquad y_2' = \beta_1 + \beta_2 y_1 + y_1^2 \pm y_1 y_2$.



(tp-lft→btm-rght):$(-1,1), (\frac{1}{4}, -1), (1,0), (0,0), (-\frac{6}{25}, -1), (0,1)$.

Scholarpedia: A bifurcation of equilibrium point in a two-parameter family of autonomous ODEs at which critical equilibrium has zero eigenvalue of (algebraic) multiplicity 2. For some nearby parameter values, system has 2 equilibria (saddle & nonsaddle) which collide & disappear via saddle-node bifurcation (see Fig1, • → 1). The nonsaddle equilibrium undergoes a Hopf bifurcation generating a limit cycle (2 → 3). This cycle degenerates into an orbit homoclinic to the saddle (3 → P) and disappears via saddle homoclinic bifurcation (P → 4).

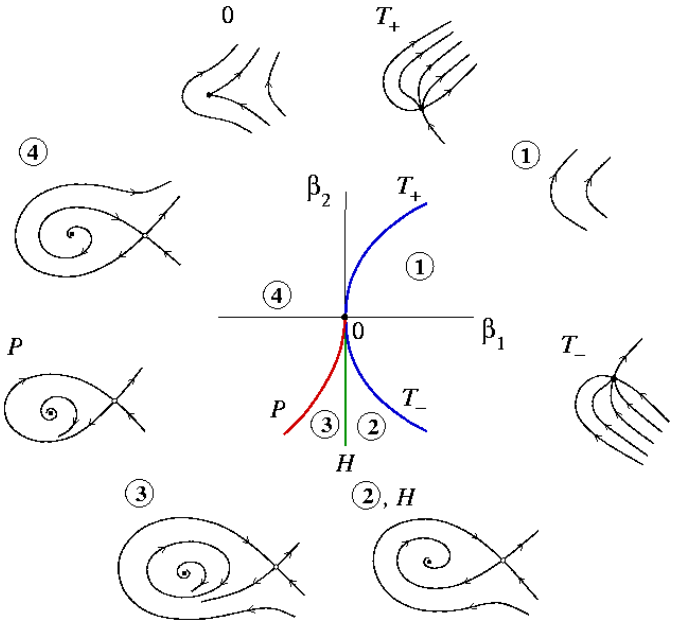


Fig1- $\dot{y}_1 = y_2$ & $\dot{y}_2 = \beta_1 + \beta_2 y_1 + y_1^2 + \sigma y_1 y_2$

Definition: Consider an autonomous system of ordinary differential equations (ODEs) $\dot{x} = f(x,\alpha),\ x \in R^n,$ (1)

depending on two parameters $\alpha \in R^2$, where f is smooth.

Suppose that at $\alpha = 0$ the system has an equilibrium $x^0 = 0$. Assume that its Jacobian matrix $A_0 = f_x(0,0)$ has a zero eigenvalue of (algebraic) multiplicity two $\lambda_{1,2} = 0$. This bifurcation is characterized by two bifurcation conditions $\lambda_1 = \lambda_2 = 0$ (has codimension two) and appears generically in two-parameter families of smooth ODEs. Generically, the critical equilibrium x^0 is a double root of the equation $f(x,0) = 0$ and $\alpha = 0$ is the origin in the parameter plane of two branches of the saddle-node bifurcation curve, an Andronov-Hopf bifurcation curve, and a saddle homoclinic bifurcation curve. Moreover, these bifurcations are nondegenerate and no other bifurcations occur in a small fixed neighborhood of x^0 for parameter values sufficiently close to $\alpha = 0$. In this neighborhood, the system has at most two equilibria and one limit cycle.

Two-dimensional BT Case

To describe the BT-bifurcation analytically, consider the system (1) with $n = 2$, $\dot{x} = f(x,\alpha), x \in \mathbb{R}^2$. If the following nondegeneracy conditions hold:

(BT.1) $a(0)b(0) \neq 0$, where $a(0)$,$b(0)$ are quadratic coefficients determined by the Taylor expansion of $f(x,0)|_{x=0}$.

(BT.2) The map $(x,\alpha) \mapsto (f(x,\alpha), trace(f_x(x,\alpha)), \det(f_x(x,\alpha)))$ is regular at $(x,\alpha) = (0,0)$,

then this system is locally topologically equivalent near the origin to the normal form:

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = \beta_1 + \beta_2 y_1 + y_1^2 + \sigma y_1 y_2,$$

where $y = (y_1, y_2)^T \in \mathbb{R}^2$, $\beta = (\beta_1, \beta_2)^T \in \mathbb{R}^2$, and $\sigma = sign(a(0)b(0)) = \pm 1$.

The local bifurcation diagram of the normal form with $\sigma = -1$ is presented in Figure 1. The point $\beta = 0$ separates two branches of the saddle-node bifurcation curve: $T_+ = \{(\beta_1,\beta_2) : \beta_1 = \frac{1}{4}\beta_2^2,\ \beta_2 > 0\}$ and $T_- = \{(\beta_1,\beta_2) : \beta_1 = \frac{1}{4}\beta_2^2,\ \beta_2 < 0\}$. The half-line $H = \{(\beta_1,\beta_2) : \beta_1 = 0,\ \beta_2 < 0\}$ corresponds to the Andronov-**Hopf** bifurcation that generates a stable limit cycle. This cycle exists and remains hyperbolic between the line H and a smooth curve $P = \{(\beta_1,\beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + O(|\beta_2|^3),\ \beta_2 < 0\}$, at which a saddle homoclinic bifurcation occurs. When the cycle approaches the homoclinic orbit, its period tends to infinity. The case $\sigma = +1$ can be reduced to the one above by the substitution $t \to -t,\ y_2 \to -y_2$. This does not affect the bifurcation curves but the limit cycle becomes unstable.

Since the kernel is one-dimensional, one sees, as expected from Lyapunov-Schmidt, a simple fold bifurcation with equilibria $y_1 = -\frac{\beta_2}{2} \pm \sqrt{\frac{\beta_2^2}{4} - \beta_1}$, $y_2 = 0$, fold location at $\beta_1 = \frac{\beta_2^2}{4}$. The trace of the linearization at these equilibria vanishes when $y_1 = 0$, that is, $\beta_1 = 0$, and the eigenvalues are purely imaginary when $\beta_2 < 0$. The frequency at the Hopf bifurcation curve converges to zero and the period of the bifurcating periodic orbits therefore tends to infinity as they approach the bifurcation point. The periodic orbits born in this Hopf bifurcation terminate on a branch of homoclinic orbits at $\beta_1 = -\frac{6}{25}\beta_2^2 + O(\beta_2^3)$, w/$\beta_2 < 0$. The periodic orbit is stable (supercritical Hopf bifurcation) in the case $\sigma = -1$, and is unstable (subcritical Hopf bifurcation) in the case $\sigma = +1$.

Additional:

Poincare Bendixson

Separatrix: boundary separating two modes of behaviour in a differential equation.

Let $\dot{x} = f(x)$ (1)

Compound Separatrix Cycle (**or Graphic**): Finite union of compatibly oriented separatrix cycles of (1).

Poincare Bendixson Basic: Suppose that $\vec{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^2$ and that (1) has a trajectory Γ with Γ^+ contained in a compact subset F of E. Then if $\omega(\Gamma)$ contains no critical point of (1), $\omega(\Gamma)$ is a periodic orbit of (1).

Poincaré Bendixson General: Suppose that $\vec{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^2$ and that (1) has a trajectory Γ with Γ^+ contained in a compact subset F of E. Also suppose (1) has only a finite number of critical points in F, it follows that $\omega(\Gamma)$ is either a critical point of (1), a periodic orbit of (1), or that $\omega(\Gamma)$ consists of a finite number of critical points $\vec{p}_1, \ldots, \vec{p}_m$; of (1) and a countable number of limit orbits of (1) whose α and ω limit sets belong to $\{\vec{p}_1, \ldots, \vec{p}_m\}$.

Devil's Staircase: Continuous function $f(x)$ on the unit interval, and C^1 off Cantor set, whose derivative is 0 almost everywhere, but it somehow magically rises from 0 to 1. Take any number x_0 in the unit interval, and express it in base 3. Chop off the base 3 expansion right after the first 1. Next, change all 2's in the expansion to 1's. This number now has only 0's or 1's in its expansion, so we can interpret it as a base 2 number. Call this new number $f(x_0)$, where $f(x)$ is the Devil's Staircase.

Maps of the Circle: (**Introduction to Applied Nonlinear Dynamical Systems and Chaos by Wiggins**)

Circle Map: C^1 orientation preserving homeomorphism of the circle, S^1, into itself: $f : S^1 \to S^1$.

The Lift of a Circle Map: Let $\Pi(x) : \mathbb{R} \to S^1$, where $\Pi(x) = e^{2\pi i x}$. The map $F : \mathbb{R} \to \mathbb{R}$ is said to be a lift of $f : S^1 \to S^1$ if $\Pi \circ F = f \circ \Pi$. "Lift F accomplishes f, but on $\mathbb{R}$."

Lifts Vary by a Constant Thm: Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism of the circle and let F_1 and F_2 be lifts of f. Then $F_1 = F_2 + k$, where k is some integer. **Proof**: Two lifts must satisfy $f \circ \Pi = \Pi \circ F_{1,2} = e^{2\pi i F_1} = e^{2\pi i F_2}$, so $F_1 = F_2 + k$.

Circle Map Lift Iterates Thm: If F is a lift of f, then F^n is a lift of f^n for $n \geq 1$. **Proof**: By definition: $\Pi \circ F = f \circ \Pi$, Therefore $\Pi \circ F^2 = \Pi \circ F \circ F = f \circ \Pi \circ F = f \circ f \circ \Pi = f^2 \circ \Pi$. And similarly for n.

Lift Arguments Chuck Integers Thm: Let $f: S^1 \to S^1$ be an orientation preserving homeomorphism of the circle and let F be a lift. Then $F(x+k) = F(x) + k$, for $k \in \mathbb{Z}$.

Periodic Functions from Lift Iterates: Let $f: S^1 \to S^1$ be an orientation preserving homeomorphism of the circle and let F be a lift of f. Then $F^n - id$ is a periodic function with period one for $n \geq 1$.

Lift Rotation Number ρ_0: For an orientation preserving homeomorphism $f: S^1 \to S^1$, with F a lift of f:
$\rho_0(F) \equiv \lim_{n\to\infty} \frac{|F^n(x)|}{n}$.

Different Lift $\rho_0 s$ **Differ by an Integer**: Let $S^1 \to S^1$ be an orientation preserving homeomorphism and let F_1 and F_2 be lifts such that $\rho_0(F_1)$ and $\rho_0(F_2)$ exist. Then, $\rho_0(F_1) = \rho_0(F_2) + k$, where $k \in \mathbb{Z}$.

Rotation Number: For $f: S^1 \to S^1$ an orientation preserving homeomorphism, with F a lift of f : the rotation number of f, denoted to $\rho(f)$ is the fractional part of $\rho_0(F)$.

Rotation Number Existence: For an orientation preserving homeomorphism $f: S^1 \to S^1$ with F, a lift of f, the rotation number exists and it is independent of x.

Periodic Points from Rotation Numbers: A rotation number is irrational if and only if f has no periodic points.

Conjugate Invariance of Rotation Number: Let f and g be orientation preserving homeomorphisms of S^1, then $\rho(f) = \rho(g^{-1}fg)$.

Rational Rotation Number: $\frac{p}{q}$

For any given initial condition, there are 3 possibilities for the orbit.

- A $\frac{p}{q}$ periodic orbit.
- A homoclinic orbit. The orbit asymptotically approaches a periodic orbit as $n \to -\infty$ and as $n \to +\infty$.
- A heteroclinic orbit. The orbit asymptotically approaches a periodic orbit as $n \to -\infty$ and a different periodic orbit as $n \to +\infty$.

Irrational Rotation Number

For any given initial condition, there are 3 possibilities for the orbit.

- An orbit that densely fills the circle.
- An orbit that densely fills a Cantor set on the circle.
- An orbit that is homoclinic to a Cantor set on the circle.

Asymptotic (nonlinear) Stability from Section 5/6 of Arnd's Notes

How do you determine asymptotic stability?

Section 5

(5.8) For linear equations, the origin is stable if $|e^{At}|$ is uniformly bounded for $t > 0$; it is asymptotically stable if $|e^{At}| \to 0$ for $t \to \infty$.

Proposition 5.11. The origin in $x' = Ax$ is asymptotically stable if and only if $Re(\sigma(A)) < 0$. In this case,

there exist constants $C, \delta > 0$ such that: $|e^{At}| \leq Ce^{-\delta t}$.

Proposition 5.12. The origin in $x' = Ax$ is stable if and only if $Re(\sigma(A)) \leq 0$ (spectrally stable) and all eigenvalues with

$Re(\lambda) = 0$ are semi-simple (linearly stable).

Section 6

Consider $x' = f(x)$, $f \in C^1$, with equilibrium at the origin (wlog), $f(0) = 0$. Set $A = Df(0)$, the linearization.

Proposition 6.1. Assume that the linearization is asymptotically stable, $Re(\sigma(A)) < 0$. Then the nonlinear equation is asymptotically stable. Moreover, there exists $C, \varepsilon, \delta > 0$ such that for all initial conditions with $|x_0| < \varepsilon$, we have: $|x(t)| \leq Ce^{-\delta t}|x_0|$. The constant $-\delta$ must be chosen larger than, but arbitrarily close to, $\max Re(\sigma(A))$.

Asymptotic Stability for Periodic Orbits. Let $\gamma(t) = \gamma(t+T),\ T > 0$ be a solution to (1). Perturbed initial condition $x_0 = \gamma(0) + y_0$. Set $y' := f(\gamma(t)+y) - f(\gamma(t)) = Df(\gamma(t))y + o(y)$. We then find $w' := A(t)y$, where $A(t)$ is T periodic.

If all Floquet exponents had a negative real part, we would be able to mimic the proof of Proposition 6.1 and conclude that $y(t) \le Ce^{-\delta t}$, and thus asymptotic stability of the periodic orbit. Since $\gamma'(t)$ contributes a Floquet multiplier $\rho = 1$, hence a zero Floquet exponent, we cannot conclude that $|e^{Bt}| \le Ce^{-\delta t}$ for $t \ge 0$. It is also clear that perturbations can in general not decay. Set for instance $y(0) = \gamma(\tau) - \gamma(0)$, such that $x(0) = \gamma(\tau)$, which is close to $\gamma(0)$ for $\tau << 1$. Then, of course, $x(t) = \gamma(t+\tau)$ and $y(t) = \gamma(t+\tau) - \gamma(t)$ does not decay but is a periodic function. In other words, perturbations do not decay. However, the periodic orbit, as a set, could still be asymptotically stable since the perturbed trajectory is of course the same periodic orbit, only shifted in time.

Asymptotic Stability via Poincaré Maps: Assume that the Floquet multipliers lie inside the unit circle with the exception of $\lambda = 1$, algebraically simple. Then the Poincaré map is a contraction near $\gamma(0)$ in a suitably defined norm.

Theorem 13 (**Asymptotic stability of periodic orbits**). Assume that the Floquet exponents λ are all contained in the negative complex half plane $\{\text{Re } \lambda < 0\}$ except for an algebraically simple Floquet exponent $\lambda = 0$. Then the periodic orbit $\Gamma = \{\gamma(t),\ 0 \le t < T\}$ is asymptotically stable. More precisely, there exist $C, \eta > 0$, a neighborhood $U(\Gamma)$, and a smooth function θ: $U \to \mathbb{R}/(T\mathbb{Z})$ such that for all $x_0 \in U$, $|x(t) - \gamma(t - \theta(x_0))| \le Ce^{-\eta t}$ for all $t \ge 0$.

Strong-Stable Manifolds: Suppose that $f(0) = 0$ and the linearization $A = Df(0)$ has an invariant "spectral splitting" at $-\eta$, for some $\eta > 0$, $\mathbb{R}^n = E^{ss} \oplus E^{wu}$, with associated projection $P^{ss}E^{ss} = E^{ss}$, $P^{ss}E^{wu} = 0$, $AP^{ss} = P^{ss}A$, $Re(\sigma(A))|_{E^{ss}} < -\eta$, $Re(\sigma(A))|_{E^{wu}} > -\eta$. We can then characterize E^{ss} as the set of x_0 such that $|e^{At}x_0| \le Ce^{-\eta t}$, for all $t \ge 0$. The strong stable manifold is $W^{ss} = \{x_0 | \Phi_t(x_0) \le Ce^{-\eta t}\}$.

Weak-Unstable Manifolds: In the previous setting of a splitting at $-\eta$, one can ask if there is a manifold tangent to E^{su}. In this case, however, the linear characterization of E^{su} as the set of x_0 such that $|e^{At}x_0| \le Ce^{-\eta t}$, for all $t \le 0$, allows for solutions with (mild) exponential growth. Such solutions are therefore not immediately characterized as solutions in a neighborhood of the origin. Nonetheless, there exists a locally invariant manifold W^{su}, of class C^ι, for some $\iota \ge 1$, given as a graph over E^{su}. Local invariance here means that solutions will stay on W^{su} as long as they remain in a neighborhood of $\mathcal{U}$ of the origin. The regularity ι is in general less than the regularity of the vector field.

Center Manifolds. The intersection of weak-unstable and weak-stable manifolds gives locally invariant manifolds tangent to a subspace of eigenvalues with $-\eta_- < \lambda < \eta_+$. Choosing $\eta_\pm << 1$, this subspace contains precisely the generalized eigenspace to eigenvalues on the imaginary axis. We note here that such a manifold tangent to the generalized eigenspace of eigenvalues on the imaginary axis exists, of class C^k for any finite k if $f \in C^k$. Asymptotic stability implies nonlinear stability, and similar statements.

Hyperbolic Implies (In)stability Theorem: Suppose that $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f \in C^1(\Omega)$, where Ω is a domain in $\mathbb{R}^n$. Also suppose that $q \in \Omega$ is an equilibrium point at which all the eigenvalues of the Jacobian matrix $f(x)$ have negative real parts. Then q is an asymptotically stable equilibrium point of f.

Suppose real parts of n_1 of the eigenvalues of the matrix $A = f_x(q)$ are positive, w/$n_1 \ge 1$ (spectrally unstable). This of course implies instability for the linearized problem and it likewise implies instability for the full, nonlinear problem.

More on Stability from a Different Source:

Structural Stability: One considers perturbations of the vector field, as opposed to perturbations of the initial data.

Let $\dot{x} = f(x),\ x \in \mathbb{R}^n$ (8.1)

Suppose this equation has an equilibrium solution q : $f(q) = 0$. If $p = q + \xi_0$ is an initial value close to q, and we set $\varphi(t,p) := q + \xi(t,p)$, then with Taylor expansion of f, equation (8.1) becomes $\dot{\xi} = A\xi + \ldots$, where $A = f_x(q)$ (8.3)

and $f_x(q)$ denotes the Jacobian matrix. From this, we get the linear part of the differential equation $\dot{\xi} = A\xi$. (8.4)

Linear Stability: The origin is asymptotically stable for the system (8.4) if the real part of every eigenvalue is negative. It is unstable if any eigenvalue has a positive real part. The existence of eigenvalues with a positive real part (spectral instability) imply instability in the linearized problem, and also for the full nonlinear problem.

Nonlinear Stability: Suppose the C^1 system (8.1) possesses the equilibrium point q and set $A := f_x(q)$. The origin is asymptotically stable if the real part of every eigenvalue of A is negative. It is unstable if any eigenvalue of A has a positive real part (spectral instability).

Conservative System: (8.1) is considered conservative if there exists a C^1 scalar function $E : \Omega \to \mathbb{R}$ which is not constant on any open set in Ω, but is constant on orbits.

Non-asymptotic in Conservative Systems: An equilibrium point q of a conservative system cannot be asymptotically stable.

Strong Minimum: A function $E : \Omega \to \mathbb{R}$ is said to have a strong minimum at q if there is a neighborhood N of q such that $E(x) > E(q)$ for every $x \in N$ except for $x = q$.

Stability from Strong Minima: Suppose q is an equilibrium point of a conservative, autonomous system and that its integral E has a strong minimum there. Then q is stable.

Linear Instability $\not\Rightarrow$ Nonlinear Instability: Let $H = p^4 + q^2$. So $\dot{p} = -\partial H_q$ and $\dot{q} = \partial H_p$, so linearized equations are $\dot{q} = 4p^3$, $\dot{p} = -2q$. $J = \begin{bmatrix} 0 & 12p^2 \\ -2 & 0 \end{bmatrix}$ Observe that the origin is an equilibrium point $(x_0, y_0) = (0,0)$. At this equilibrium we have: $J_0 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$ with $\lambda = 0$ (so not hyperbolic), which is obviously spectrally stable. Set $A := (J_0 - \lambda \mathbb{I}) = J_0$. Calculating the generalized eigenvectors: $A^2 v_2 := A^2 \langle 1\ 0 \rangle = 0$ and $A v_2 = \langle 0, -2 \rangle =: v_1$ has solutions increasing linearly with time ($\langle q, p \rangle = c_1 \langle 0, -2 \rangle + c_2 (\langle 0, -2 \rangle t + \langle 1, 0 \rangle)$), so the origin is unstable for the linearized system. However, clearly the Hamiltonian has a strong minimum there, so it is nonlinearly stable.

Linear Stability $\not\Rightarrow$ Nonlinear Stability: $H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) - \frac{1}{2} p_2 (q_1^2 - p_1^2) - q_1 q_2 p_1$ (Cherry Hamiltonian) The EOM are $\dot{q}_1 = p_1 + p_1 p_2 - q_1 q_2$, $\dot{p}_1 = -q_1 + q_1 p_2 + p_1 q_2$, $\dot{q}_2 = -2p_2 + \frac{1}{2}(p_1^2 - q_1^2)$, $\dot{p}_2 = 2q_2 + q_1 p_1$.

The linearized system can be read from the above. It possesses periodic solutions. However, for arbitrary T, one can verify that a solution to the full system is also: $p_1 = \sqrt{2}\,\frac{\sin(t-T)}{t-T}$, $q_1 = \sqrt{2}\,\frac{\cos(t-T)}{t-T}$, $q_2 = \frac{\cos(2(t-T))}{t-T}$, $p_2 = \frac{\sin(2(t-T))}{t-T}$. Observe that if $T > 0$, then these solutions become unbounded in finite time.

Is it true or false that you have a stable equilibrium if and only if the real part of the eigenvalues of the linearization are strictly less than zero:

Stable Equilibrium $\Leftrightarrow \mathrm{Re}(\lambda_i) < 0$?

No: While it is true that $\mathrm{Re}(\lambda_i) \Rightarrow$ Stable Equilibrium. The inverse implication is incorrect.

For example, take $x' = -x^3$. Observe that $x = 0$ is a fixed point, and $\forall x < 0$, we have $x' > 0$, and $\forall x > 0$, we have $x' < 0$, so the fixed point is stable. However, note that the linearization at the fixed point $x = 0$ is $\partial_x x^3|_{x=0} = 3(0)^2 = 0$. Therefore, we have $\lambda = 0$ and $Re(\lambda) = 0$. So, Stable Equilibrium $\not\Rightarrow \mathrm{Re}(\lambda_i) < 0$.

Nonlinear Circle Map: $\theta_{n+1} = \theta_n + \Omega + \frac{K}{2\pi} \sin(2\pi\theta_n) \pmod 1$, where K is the coupling strength which determines the degree of nonlinearity, and Ω is an externally applied driving frequency.

Arnold Tongues: Phase-locked or mode-locked region in a driven (kicked) weakly-coupled harmonic oscillator. If $K = 0$ and Ω is rational, then it is known that the map winding number, defined as $\Omega = W \equiv \frac{p}{q}$, and implies a periodic trajectory, since θ_n will return to the same point (at most) every q map orbits. If Ω is irrational, then the motion is quasiperiodic. If K is nonzero, then the motion may be periodic in some finite region surrounding each rational Ω. This execution of periodic motion in response to an irrational forcing is known as **mode locking**.

If a plot is made of K vs. Ω with the regions of periodic mode-locked parameter space plotted around rational Ω values (map winding numbers), then the regions are seen to widen upward from 0 at $K = 0$ to some finite width at $K = 1$. The region surrounding each rational number is known as an **Arnold tongue**. At $K = 0$, the Arnold tongues are an isolated set of measure zero. At $K = 1$, they form a Cantor set of dimension dim ≈ 0.08700. For $K > 1$, the tongues overlap, and the

circle map becomes noninvertible.

Mode locking: For small to intermediate values of K ($0 < K < 1$), and certain values of Ω, the map exhibits a phenomenon called mode locking or phase locking. In a phase-locked region, the values θ_n advance essentially as a rational multiple of n, although they may do so chaotically on the small scale. The limiting behavior in the mode-locked regions is given by the rotation number. $W = \lim_{n\to\infty} \frac{\theta_n}{n}$, which is also sometimes referred to as the map winding number.

The phase-locked regions, or Arnold tongues form V-shaped regions that touch down to a rational value $\Omega = \frac{p}{q}$ in the limit of $K \to 0$. The values of (K, Ω) in one of these regions will all result in a motion such that the rotation number $W = \frac{p}{q}$. For example, all values of (K, Ω) in the large V-shaped region in the bottom-center of the figure correspond to a rotation number of $W = \frac{1}{2}$. One reason the term "locking" is used is that the individual values θ_n can be perturbed by rather large random disturbances (up to the width of the tongue, for a given value of K), without disturbing the limiting rotation number. That is, the sequence stays "locked on" to the signal, despite the addition of significant noise to the series θ_n.

There is a mode-locked region for every rational number $\frac{p}{q}$. It is sometimes said that the circle map maps the rationals, a set of measure zero at $K = 0$, to a set of non-zero measure for $K \neq 0$.