

# Classical Mechanics: Class Notes

Below you will find the notes I took while auditing Leonard Susskind's Stanford University Lectures on Classical Mechanics (via YouTube). I make no guarantee as to the accuracy of these notes. Since I wrote them (while auditing the class), I have not edited them. And, they no doubt need editing! So, reader beware. I also only began taking notes halfway through lecture 3. Relatedly, you'll probably notice is that I became a much better notetaker as the class went on. Please give me feedback on my website (<http://curiouschimp.com/wp/index.php/2015/06/07/classical-mechanics-class-notes/>) regarding errors or edits I can make to the document. This will not only help anyone else who might use my notes, but will help my memory to stay fresh regarding this topic. Thanks!

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Leonard Susskind's Modern Physics course concentrating on Classical Mechanics.

Recorded 2007 at Stanford University.

This Stanford Continuing Studies course is the first of a six-quarter sequence of classes exploring the essential theoretical foundations of modern physics. The topics covered in this course focus on classical mechanics. Leonard Susskind is the Felix Bloch Professor of Physics at Stanford University.

Complete playlist for the course: <https://goo.gl/DiwCv2>

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Associated with...PHYSICS 110: Advanced Mechanics (PHYSICS 210)

Lagrangian and Hamiltonian mechanics. Principle of least action, Euler-Lagrange equations. Small oscillations and beyond. Symmetries, canonical transformations, Hamilton-Jacobi theory, action-angle variables. Introduction to classical field theory. Selected other topics, including nonlinear dynamical systems, attractors, chaotic motion.

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## Lecture 3, (about halfway through)

Definition of the Lagrangian: The kinetic energy minus the potential energy.

$$T - U = \mathcal{L}$$

$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m_i \dot{x}_i = p_i$ , momentum of a particular particle with respect to a particular direction.

$\frac{d}{dt} m_i \dot{x}_i = -\frac{\partial U}{\partial x_i} = m_i \ddot{x}_i$ , is the force ( $F = ma$ ) where  $x_i$  specifies the particle and a particular direction.

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Two particles on a line, interacting with a force governed by a potential energy.

Particle 1 ( $x_1$ ) has mass  $m_1$ , and particle 2 ( $x_2$ ) has mass  $m_2$ .

The potential energy ( $U$ ) depends upon the position of the two particles. We are going to assume translational invariants, in other words, it doesn't matter where the particles are, but rather, only the distance between them (translation symmetry).

Velocity is also often symmetric. It doesn't matter where you choose to locate the origin in your system, the velocities are constant. For example, if your location function is  $x(t)$ , adding a constant  $c$ , doesn't change the derivative.  $v = (x(t) + c) = x'(t)$ .

For our original problem, the kinetic energies are  $\frac{1}{2}m_1 \dot{x}_1^2 + \frac{1}{2}m_2 \dot{x}_2^2$ .

And since the potential energy only depends upon the distance between the two particles...

$$\mathcal{L} = T - U = \left( \frac{1}{2}m_1 \dot{x}_1^2 + \frac{1}{2}m_2 \dot{x}_2^2 \right) - U(x_1 - x_2).$$

Particle 1:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} + \frac{\partial \mathcal{L}}{\partial x_1} = 0,$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = -\frac{\partial \mathcal{L}}{\partial x_1},$$

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$$\frac{dp_1}{dt} = -\frac{\partial U}{\partial x_1} \text{ (since the only term that depends upon } x_1 \text{ is } U)$$

Particle 2:

$$\frac{dp_2}{dt} = -\frac{\partial U}{\partial x_2} \text{ (since the only term that depends upon } x_2 \text{ is } U)$$

What is the difference between  $\frac{\partial U}{\partial x_1}$  and  $\frac{\partial U}{\partial x_2}$  (the forces of the two particles)? They are equal and opposite.

What we find is that momentum conservation is a result of translation invariance in a simple system like this, and that the equations used are derived from the concept of least action.

- Symmetries says that the potential energy ( $U$ ) only depends on  $x_1$  and  $x_2$ .
- From that, it follows that the force on particle 1 is equal to an opposite the force on particle 2.
- And from that follows the conservation of linear momentum

A symmetry is defined by an active operation that you can do on the system (for example, translating it), which does not change the value of the action (or the Lagrangian).

2D: The surface of the earth

$$\text{Kinetic energy: } T = \frac{1}{2}m \dot{x}^2 + \frac{1}{2}m \dot{y}^2$$

$$\text{Potential energy: } U = (\text{height})(\text{mass})(\text{gravity}) = y \cdot m \cdot g$$

$$-\frac{\partial U}{\partial y} = -mg, \text{ (the minus sign is to indicate the force is downward)}$$

$$\mathcal{L} = T - U = \left( \frac{1}{2}m \dot{x}^2 + \frac{1}{2}m \dot{y}^2 \right) - ymg$$

$$\frac{\partial \mathcal{L}}{\partial x} = m \dot{x}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} m \dot{x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = m \dot{y}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt} m \dot{y} = -mg = -\frac{\partial U}{\partial y}, \text{ (the vertical acceleration changes with respect to gravity)}$$

Polar coordinate:

The trajectory of a particle is determined by two functions,  $r(t)$  and  $\theta(t)$ .

$$\text{Kinetic energy: } T = \frac{1}{2}m\dot{r}^2$$

Velocity along the  $r$  direction, we will call  $v_r = \dot{r}$ . Similarly, velocity along the  $\theta$  direction, we will call  $v_\theta = r \dot{\theta}$ .

$$\text{Kinetic energy: } T = \frac{1}{2}m \left[ \dot{r}^2 + r^2 \dot{\theta}^2 \right].$$

Potential energy: Central force

We are going to have the potential energy depend only upon  $r$ . This means it will only depend upon its distance from the origin. This means it will be invariant if we rotate the axis. Therefore, it will have a rotational symmetry.

$$\mathcal{L} = T - U = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = m \dot{r}, \text{ (the momentum conjugate to } r)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial r} = m \ddot{r} = -\frac{\partial U}{\partial r}, \text{ (the radial component of acceleration)}$$

$mr^2 \dot{\theta}$ , (angular momentum, or the conical momentum conjugate of  $\theta$ )

$$mr^2 \dot{\theta} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt} mr^2 \dot{\theta} = 0, \text{ (conservation of angular momentum)}$$

So, solving for  $\dot{\theta}$ ,  $\dot{\theta} = \frac{k}{mr^2}$ , for some constant  $k$ . So, as the radius decreases, the angular momentum

increases (think about how a spinning ice skater speeds up as they bring their arms closer to their body).

To determine the action of a system, we first start out by

Canonical momentum conjugate of the coordinate, for each coordinate ( $i = r, r\theta, x, y, z$ , etc.).

This is the derivative of the Lagrangian with respect to the time derivative of the coordinate ( $\dot{q}_i$ )  $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \Pi_i$ .

The Euler/Lagrange equation is the derivative of the canonical momentum conjugates ( $\frac{d}{dt} \Pi_i = \frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial \mathcal{L}}{\partial q_i}$ )

## Lecture 4

So, in our previous example,  $\Pi_r = m \dot{r}$ .

The Euler/Lagrange equation:  $\frac{d}{dt} \Pi_r = m \ddot{r} = \frac{\partial \mathcal{L}}{\partial r}$ . But,  $\frac{\partial \mathcal{L}}{\partial r}$  where  $\mathcal{L} = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$  is  $mr \dot{\theta}^2 - \frac{\partial U}{\partial r}$ .

So,  $m \ddot{r} = mr \dot{\theta}^2 - \frac{\partial U}{\partial r}$ . However,  $mr \dot{\theta}^2$  depends upon the particle moving in the  $\theta$  direction, and because it is squared, it is always a positive (pushing outward radially) force (centrifugal force).

Remember that we are still working with the partial to the radial force, and are therefore only calculating forces in the radial direction.

Now let us calculate the forces in the  $\theta$  direction.  $\Pi_\theta = mr^2 \dot{\theta}$ .  $\frac{d}{dt} \Pi_\theta = \frac{d}{dt} (mr^2 \dot{\theta}) = \frac{\partial \mathcal{L}}{\partial \theta}$ . But,  $\frac{\partial \mathcal{L}}{\partial \theta}$  where  $\mathcal{L} = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$  is 0. So, the system is rotationally invariant, and it has rotational

symmetry. The derivative with respect to time of the angular momentum ( $mr^2 \dot{\theta}$ , which is usually notated as  $L$ ) is 0, which means that the angular momentum must be a constant. This must mean that there is a conservation law (because it only depends upon initial conditions  $r, \theta, \dot{\theta}, \dot{r}$ , unlike the forces in the radial direction. This allows us to solve for  $\theta$ ,  $\theta = \frac{L}{mr^2}$ .

This allows us to revisit the equation for  $\frac{d}{dt} \Pi_r = m \ddot{r} = mr \dot{\theta}^2 - \frac{\partial U}{\partial r} = mr \left( \frac{L}{mr^2} \right)^2 - \frac{\partial U}{\partial r} = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r}$ . Or  $m \ddot{r} = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r}$ . So, we now have an equation which includes the term for centrifugal force in terms of angular momentum. The equation also tells us that the force is smaller when the radius is large.

So this was both an example of how to use the Lagrange equations, and to give you an example of the symmetry and how it acts to create a conserved quantity (angular momentum).

Let's say you have a function with a lot of variables, and your job is to minimize it.

$F(\alpha_i)$  where  $\delta F = 0$ , which means if you change  $\alpha_i$  by a small degree, the output of the function does not change to first order. To calculate this, you sum up all of the differentials of the function with respect to the different variables...

$$\sum_i \frac{\partial F}{\partial \alpha_i} \delta \alpha_i = 0 \Leftrightarrow \delta F = 0.$$

With the Lagrangian, this might equate to the small change of trajectory of the system. In this case, we are looking for a Lagrangian in which the above summation is zero for small changes in the trajectory.

## Relationship between symmetries and conservation laws...

Examples given up until now include rotational and translational symmetry.

Symmetry means that you can change the coordinate system in some way, and the action is unchanged. This means that the action does not depend upon the location of individual particles of the system, but rather the relative locations of these particles for translational symmetry for example. In particular, let us look at infinitesimal symmetry, which we can build up into a large symmetry. So, as a result we can assume that a large symmetry exists when we have shown an infinitesimal symmetry (in other words, working to first order in small quantities). Let us change the coordinate  $q_i$  by a small amount  $\varepsilon f_i(q)$ ,  $q_i \rightarrow q_i + \varepsilon f_i(q)$ . In the situation,  $f_i$  depends upon all of the  $q_s$ . For example, if this is a rotational change, a particle's change in location under this change in coordinate depends upon its location ( $q_s$ ). If its location along the X axis decreases in quadrant I (for a counterclockwise change), its location would increase in quadrant IV. Saying we have a symmetry, though, says that the action is invariant under this change.

For example, if we look at rotation with Cartesian coordinates, and we imagine a particle that is located on the x-axis, we find that a small rotational change causes the following change in the x-coordinate:  $\delta x = -\varepsilon y$ , and the change in the y-coordinate is:  $\delta y = \varepsilon x$ . In our equation above, this would equate to  $f_x = -y$  and  $f_y = x$ .

Our goal is to prove that there is a conserved quantity when there is a symmetry. Let us imagine a point moving through time and also through dimensions of space. We have a beginning and end point, timewise. And, let us assume that for all infinitesimal changes (except changing the starting and ending point), the action does not change  $\delta A = 0$ .

Now, let us imagine that you did change the starting and endpoint as a result of altering the system as a result of a translation, a rotation, or some other coordinate change in space. This type of change results in no change in the action. This is not due to the principle of least action, however, but rather due to the fact that we have defined there being a symmetry.

Let's compute

$$\text{it: } A = \int \mathcal{L} dt \quad \delta A = \sum_i \int \left( \frac{\partial \mathcal{Q}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \sum_i \int \left( \frac{\partial \mathcal{Q}}{\partial q_i} \delta q_i - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i \right) = \sum_i \int \left( \frac{\partial \mathcal{Q}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \delta q_i.$$

However, this is not necessarily the right answer because we have also displaced the endpoints. There is an endpoint contribution coming from the integration by parts. It happens because  $\delta q_i$  is not zero at the endpoints. So it is not a "legal" alteration from the standpoint of the principle of least action. So, in order to solve the equation correctly, we have to add back in the missing term  $\frac{\partial \mathcal{Q}}{\partial q_i} \delta q_i|_{t_1}^{t_2}$ , the difference of the quantity at the two endpoints.

$\delta A = \sum_i \int \left( \frac{\partial \mathcal{Q}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \delta q_i + \sum_i \frac{\partial \mathcal{Q}}{\partial q_i} \delta q_i|_{t_1}^{t_2}$ . Because we assumed that the original equation was a solution, we know that the part of the equation within the parentheses is equal to zero, and therefore can be ignored. Also, because we know we have a symmetry, we know that  $\delta A = 0$ . So,  $\sum_i \frac{\partial \mathcal{Q}}{\partial q_i} \delta q_i|_{t_1}^{t_2} = 0$ . Saying that the difference between this quantity at the beginning and the end is zero, is the same as saying that the quantity is conserved. Therefore, if we start with a symmetry, we end up with a conservation law.

The conserved quantity is  $\sum_i \frac{\partial \mathcal{Q}}{\partial q_i} \varepsilon f_i(q)$ . And since it is conserved, and therefore the derivative is 0, we do not need the  $\varepsilon$ , as it is still equal to zero without it. Therefore,  $\sum_i \frac{\partial \mathcal{Q}}{\partial q_i} f_i(q)$  is conserved and

$$\frac{d}{dt} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} f_i(q) = 0.$$

However, just because a system is invariant under a transformation, does not mean there is a conservation associated with it. This is only true when this invariance comes about by way of the least action principle. Symmetry, plus the assumption of least action leads to a conservation law.

Written another way,  $\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} f_i(q) = \sum_i \Pi_i \delta q_i = \sum_i \Pi_i f_i(q)$  is conserved (Noether's Charge). Physicists tend to refer to anything that is conserved as a charge (momentum is a charge, and energy is considered a charge, etc.).

## Let's do an example...

Let's assume we have a whole bunch of particles. The system is invariant in regards to translation. There are several symmetries (one for each set of directions), but we are going to choose to look at the symmetry along the x-axis. Let us use the index  $i$  to label the particular particle where referencing.  $\delta x_i$  is therefore the change in the x-coordinate of the  $i$ th particle.  $\delta x_i = \varepsilon$ .  $\delta y_i = \delta z_i = 0$  (for this x-coordinate symmetry).  $f = 1$  (for all of the  $f$ s associated with the x-coordinate of the  $i$ th particle).

Now let us calculate the Noether's Charge:

$$\sum_i \Pi_x f = \sum_i \Pi_x (1) = \sum_i \Pi_x = \sum_i m_i \dot{x}_i.$$

Notice that we did not need to know anything about the forces, only that the potential energy does not change when you translate the whole system.

Now let us work out this for angular momentum in Cartesian coordinates...

Imagine a particle moving through time in a plane (x-coordinate and y-coordinate). In this system, we will examine one particle, and examine it when we alter the system by rotating it by an amount  $\varepsilon$  (measured in radians).

Change in the x-coordinate is:  $\delta x = -\varepsilon y = \varepsilon f_x$ , and the change in the y-coordinate is:  $\delta y = \varepsilon x = \varepsilon f_y$ .

So,  $f_x = -y$  and  $f_y = x$ .

Now let us calculate the Noether's Charge:

Let's rewrite  $\Pi_x = P_x$

$$\sum_i [\Pi_x f_x + \Pi_y f_y] = \sum_i [P_x(-y) + P_y(x)] = \sum_i [xP_y - yP_x] = L \text{ (angular momentum).}$$

To come upon this conclusion, we did not need to know many things about the system. Only that the action did not change under rotation.

Typically, a system that is not under the influence of an external factor (a close to system) will be invariant under time translation (time translation invariants). In other words, you can alter the starting and ending time by the same amount, and the trajectories will be conserved. This is not true if there is an external factor which is exhibiting a change in the forces between the particles in the system as a function of time.

Let us again imagine a particle moving through one-dimensional space (the blackboard) and through time. Imagine we already have a solution for the trajectory. Now, imagine shifting each point along the trajectory upward (forward in the time direction) by an equal amount.  $q(t) \rightarrow q(t - \varepsilon)$ .

While we can think of the shift as being upward in the time direction, let us instead think about it in

terms of shifts in the x-coordinate.  $\delta q = -\frac{dq}{dt} \varepsilon = -\dot{q} \varepsilon$ . Visually, this will appear as a trajectory which has been shifted upward on the blackboard, or alternatively shifted to the left of the blackboard. We're making the assumption that the given trajectory is a solution to the equation, and therefore that the shifted trajectory is also a solution (because of symmetry under time translation). That tells us that the action of the new trajectory is the same as the action of the old trajectory.

Let us calculate the change in the action from the old trajectory to the new trajectory (it should be zero). Let us also cut up the trajectories into three parts. The first part  $B = t_0 \rightarrow t_a$  this from the beginning of the initial trajectory, to the beginning of the new trajectory (these are separated in time due to the translation). The second part  $t_a \rightarrow t_b$  is from the beginning of the new trajectory, to the end of the old trajectory. And the last part  $A = t_b \rightarrow t_f$  is from the end of the old trajectory, to the end of the new trajectory.

$$\int_a^b \left[ \frac{\partial \mathcal{L}}{\partial q} (-\dot{q} \varepsilon) + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right] dt + A - B$$

$$\int_a^b \left[ \frac{\partial \mathcal{L}}{\partial q} \delta q - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right] dt + \delta q \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} + A - B = \int_a^b \left[ \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right] dt + \delta q \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} + A - B$$

where  $\delta q \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b}$  represents the difference in the action which is contributed to by way of the new initial and final endpoint. The need for this term is a result of our having integrated by parts (it is because  $\delta q \neq 0$  at these points, and so their contributions to the overall action must be calculated).

Because we know the initial equation satisfied the equation of motion, we know that the part of the equation above which relates to the new motion that is identical to the motion of the original system (but now shifted timewise), still satisfies the equations of motion, and that therefore this part of the equation must equal zero  $\int_a^b \left[ \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right] dt = 0$ .

So, because we know the total action is also zero,  $0 = \delta q \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} + A - B = -\varepsilon \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} + A - B$ .

Now we have to figure out how to calculate  $A$  and  $B$ . This means working out the Lagrangian for that initial piece of trajectory ( $B$ ), and the final piece ( $A$ ). For the final piece, the time involved is  $\varepsilon$ , and since this is assumed to be a very small period of time, the Lagrangian can be estimated as  $\mathcal{L}(t_b)\varepsilon = A$ .

Similarly,  $\mathcal{L}(t_a)\varepsilon = B$ .

$$\text{So, } 0 = -\varepsilon \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} + \mathcal{L}(t_b)\varepsilon - \mathcal{L}(t_a)\varepsilon$$

$$\mathcal{L} = \mathcal{L}(t_b) - \mathcal{L}(t_a).$$

So,  $0 = \mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b}$ , and  $\mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_a}^{t_b} = \mathcal{L} - \dot{q} \Pi$  is conserved. If we assume this to be calculated over all particles, it can also be written as  $\mathcal{L} - \sum_i \dot{q}_i \Pi_i$ . If a quantity is conserved, so is its negative, so  $-\mathcal{L} + \sum_i \dot{q}_i \Pi_i$  is conserved, and is the form normally used in such equations, and is usually notated with the shorthand  $H$  (the Hamiltonian, or the energy of the system). Let's work it out for an example, and show that it actually is the energy of the system...

$$\frac{1}{2} m \dot{x}^2 - U(x) = \mathcal{L}$$

The canonical momentum with respect to  $x$ :  $m \dot{x} = \Pi$

$$\text{So, } \dot{q} \Pi = \dot{x} \Pi = \dot{x} (m \dot{x}) = m \dot{x}^2$$

Then, we have to subtract the Lagrangian:  $H = m \dot{x}^2 - \left( \frac{1}{2} m \dot{x}^2 - U(x) \right) = \frac{1}{2} m \dot{x}^2 + U(x)$ .

So, the definition of energy in mechanics is the quantity which is conserved as a consequence of time translation invariance.

As long as your system is time translation invariant, and assuming it comes from a principle of least action, there is always a conserved energy.

## Lecture 5

Examples of how much easier it is to work with a Lagrangian than to use the traditional physics equations. This is because, in order to work with  $F = ma$ , one has to determine accelerations which are second derivatives. The Lagrangian only requires velocities. If you can calculate the velocities of a system, you can calculate the kinetic energy  $\frac{1}{2}mv^2$ . With the locations of the particles, you can also work out the potential energy, and then ultimately the equations of motion.

### Simple Pendulum:

The pendulum will consist of a rigid rod with a ball at the end.

The ball's mass at the end of the pendulum we will label  $m$ .

The angle relative to the vertical we will call  $\theta$ .

The potential energy is a function of  $\theta$ , and the kinetic energy is the velocity of the ball.

Let us call the length of the pendulum  $r$ , which does not change.

The horizontal distance from the vertical to the ball at any moment of time can be calculated as:  $r \sin \theta$ .

The vertical distance can be calculated as  $r \cos \theta$ .

In order to calculate the velocity of the ball, we must take the derivative of the location. This means taking the derivative of the x-coordinate, and the y-coordinate with respect to time.

$r$  is not a function of time, but  $\theta$  is, therefore when we are differentiating, we treat  $r$  as a constant and  $\theta$  as a function of  $t$ .

$$v = \left( \frac{d}{dt} r \sin \theta, \frac{d}{dt} r \cos \theta \right) = \left( r \cos \theta \dot{\theta}, -r \sin \theta \dot{\theta} \right)$$

Kinetic energy

$$T = \frac{m}{2} (\text{sums of the squares of the components of the velocities}) = \frac{m}{2} \left( (r \cos \theta \dot{\theta})^2 + (-r \sin \theta \dot{\theta})^2 \right) \\ = \frac{m}{2} r^2 \dot{\theta}^2 (\cos^2 \theta + \sin^2 \theta) = \frac{m}{2} r^2 \dot{\theta}^2 .$$

We are choosing to define the potential energy as zero when the pendulum is in the vertical position. The choice can be made for the potential energy to be zero anywhere in the cycle, and it doesn't change our calculations.

Potential energy  $U = m \cdot g \cdot h = mgr(-\cos \theta)$  where  $g$  is the force of gravity.

$$\mathcal{L} = \frac{mr^2}{2} \dot{\theta}^2 - (-mgr \cos \theta)$$

$$\Pi_{\theta} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \dot{\theta}$$

The equation of motion:  $\frac{d}{dt} \Pi_{\theta} = U$

$$\frac{d}{dt} mr^2 \dot{\theta} = -mgr \cos \theta$$

$$mr^2 \ddot{\theta} = -mgr \cos \theta$$

$$r \ddot{\theta} = -g \cos \theta$$

Now let us calculate the Hamiltonian (the total energy of the system):

$$H = \Pi_{\theta} \dot{\theta} - L = mr^2 \dot{\theta}^2 - \left( \frac{mr^2}{2} \dot{\theta}^2 + mgr \cos \theta \right) = \frac{mr^2}{2} \dot{\theta}^2 - mgr \cos \theta.$$

Since  $mgr$  is the maximum potential energy (when  $\theta = 0$ ), and  $-mgr$  is the minimum potential energy (when  $\theta = 180$ ), it takes a kinetic energy of  $2mgr$  (by pushing the ball), to move the ball from it stationary position all the way to the top of its cycle. Any more than this amount will cause it to make a full cycle, and any less will cause it to oscillate back-and-forth.

And the point of all this was to show that this calculation is much easier with the Lagrangian than it would be with the traditional Newtonian equation  $F = ma$ .

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## Example: Double pendulum

Like the previous example, there is a pivot point to which a ball is connected with a rigid rod (with mass  $m$ , and length  $r$ , respectively). However, connected to the first ball is another rigid rod with a second ball on its end (and for simplicity, the second rod and ball have the same mass and length as the first).

Much more complicated.

In appearance, it is chaotic. It would take hours to work out the traditional formulas for the system, while the Lagrangian is much easier.

The first thing you should do with a problem like this is choose coordinates. With experience, you will get better at using the most useful set of coordinates. Let us call the angle of the first Rod from the vertical  $\theta$ . For the angle of the second Rod, one could easily choose either the angle relative to the first Rod, or the angle relative to the vertical. For this example, let's choose the angle relative to the vertical and call it  $\phi$ .

Let's calculate the kinetic and potential energy...

Kinetic energy for the first ball:  $v_1 = \frac{mr^2}{2} \dot{\theta}^2$  (see the previous example)

Kinetic energy for the second ball: This depends upon the movement of the first ball. Even if the angle of the second ball were not moving, the second ball WOULD still be moving due to the motion provided by the movement of the first ball (which serves as the second ball's pivot point). So, our calculation for the kinetic energy of the second ball requires we add to velocities, the velocity of the second ball and the velocity of the first...

The X component of the second ball's velocity includes the X component of the first ball's velocity plus the X component of the second ball's velocity.

$$\begin{aligned} v_2 &= \left( r \cos \theta \dot{\theta} + r \cos \phi \dot{\phi}, -r \sin \theta \dot{\theta} - r \sin \phi \dot{\phi} \right) \\ &\frac{1}{2} m \left[ \left( r \cos \theta \dot{\theta} + r \cos \phi \dot{\phi} \right)^2 + \left( -r \sin \theta \dot{\theta} - r \sin \phi \dot{\phi} \right)^2 \right] \\ &= \frac{1}{2} m \left[ r^2 \dot{\theta}^2 + 2r^2 \cos \theta \cos \phi \dot{\theta} \dot{\phi} + r^2 \dot{\phi}^2 + 2r^2 \sin \theta \sin \phi \dot{\theta} \dot{\phi} \right] \\ &= \frac{1}{2} m r^2 \left[ \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} (\cos \theta \cos \phi + \sin \theta \sin \phi) \right] \\ T = v_1 + v_2 &= \left( \frac{mr^2}{2} \dot{\theta}^2 \right) + \frac{1}{2} m r^2 \left[ \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} (\cos \theta \cos \phi + \sin \theta \sin \phi) \right] = \frac{1}{2} m r^2 \left[ 2 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \right. \\ &= \frac{1}{2} m r^2 \left[ 2 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right]. \end{aligned}$$

And now the potential energy...

$$U = -2mg \cos \theta - mg \cos \phi.$$

So the Lagrangian is...

$$\begin{aligned} \mathcal{L} = T - U &= \frac{1}{2} m r^2 \left[ 2 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right] - (-2mgr \cos \theta - mgr \cos \phi) \\ &= \frac{1}{2} m r^2 \left[ 2 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right] + mgr(2 \cos \theta + \cos \phi). \end{aligned}$$

Because there is a favored direction for gravity in this example (downward), there is not a rotational symmetry. And therefore, no related conserved quantity.

However, if we were to do this experiment out in space, where there was no gravity, we would find a rotational symmetry. This would allow us to delete the potential energy ( $mgr(2 \cos \theta + \cos \phi)$ ) from the equation giving us...

$$\mathcal{L} = \frac{1}{2} m r^2 \left[ 2 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right]$$



Rotational symmetry can be seen when you add a small amount of angle measure to both  $\theta$  and  $\phi$ .  $\theta \rightarrow \theta + \varepsilon$  and  $\phi \rightarrow \phi + \varepsilon$ . This is similar to the previous example in which  $q \rightarrow q + \varepsilon f(q)$ , except  $f(q) = 1$ . Let us calculate the angular momentum, or the Noether's charge (which does not change with time).  $\sum_i \Pi_i f_i$  where  $\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$  and where  $f_i = 1$ .

$$\sum_i \Pi_i f_i = \Pi_\theta + \Pi_\phi.$$

$$\Pi_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2} m r^2 \left[ 4 \dot{\theta} + 2 \dot{\phi} \cos(\theta - \phi) \right]$$

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m r^2 \left[ 2 \dot{\phi} + 2 \dot{\theta} \cos(\theta - \phi) \right]$$

$$\sum_i \Pi_i f_i = \Pi_\theta + \Pi_\phi = m r^2 \left[ 2 \dot{\theta} + \dot{\phi} + (\dot{\phi} + \dot{\theta}) \cos(\theta - \phi) \right] = L \text{ (angular momentum, and it doesn't change with time)}$$

Now let us calculate the equations of motion (for  $\theta$ )...

$$\frac{d}{dt} \Pi_\theta = \frac{d}{dt} \left( \frac{1}{2} m r^2 \left[ 4 \dot{\theta} + 2 \dot{\phi} \cos(\theta - \phi) \right] \right) = -m r^2 \dot{\phi} \sin(\theta - \phi).$$

## Harmonic Oscillator

The harmonic oscillator is similar to the previous example, especially in the case when the pendulum has been given a small amount of kinetic energy.

We can use this knowledge to think about the new equations.

Let us take a look at the potential energy:  $-m g r \cos \theta$ , and if we ignore the constants, it looks like  $-\cos \theta$ , which is a wave function in the Cartesian coordinate plane. Its minimum is at  $\theta = 0$  (when it is hanging straight down). When a function is minimized at a particular point, it indicates that the derivative of that function at that point is 0. The function is also well approximated near that point by a parabola. In other words, something that is proportional to  $\theta^2$ . So, when  $\theta$  is small, the potential energy can be approximated by the minimum potential energy ( $-m g r$ , see the previous notes on the simple pendulum regarding this) plus this parabolic term...  $-m g r + \frac{1}{2} m g r \theta^2$ .

This would make our Lagrangian (we can leave out the constant term as the Lagrangian is always differentiated causing any constant term to vanish)...

$$\mathcal{L} = \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{1}{2} m g r \theta^2.$$

The Takeaway: When the potential energy is a quadratic in the coordinate, it's a harmonic oscillator.

Let's imagine a spring connected to a mass  $m$  (imagine the spring stretching horizontally, along an  $x$ -axis, with the mass attached, I've heard this model described as being a mass which is a block of ice sliding frictionless on a surface of ice). If you stretch it or compress it, the mass will begin to oscillate back-and-forth around the equilibrium point (where  $x = 0$ ). Similar to the previous example, any work used to displace the mass from the equilibrium point can be expressed as being proportional to  $x^2$ .

$U = \frac{1}{2} k x^2$  where  $k$  creates the arbitrary proportionality, and is referred to as the spring constant. By convention, the equation includes  $\frac{1}{2}$  to make the calculations work out better under differentiation.

$$\mathcal{L} = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Remember that the force is equal to  $F = \frac{\partial U}{\partial x} = -kx$ . The minus sign means that the force is a "restoring force". If you stretch the spring such that  $x > 0$ , the force is negative (it pulls you back). And vice versa. The fact that the force is proportional to  $x$  is referred to as Hooke's law (Newton's best friend).

Imagine attempting to use a Taylor series to try to approximate the shape of a harmonic function near the bottom of one of the curves. Additionally, we are planning to differentiate this series. Therefore, the constant term doesn't matter and can be ignored.  $ax^1$  can also be ignored since we know  $x = 0$  once we

have differentiated (since we are at the minimum). So, any such series only contains terms we are interested in starting at  $bx^2 + cx^3 + \dots$ . Indeed, for small oscillations, you can even ignore all but the  $bx^2$  term (unless it is equal to zero).

The energy  $H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ . Because both of these terms are positive, conservation of energy will not allow the kinetic energy in the system to go above a limiting amount. Therefore, the oscillations can only be so large.

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ , To work out the equations of motion for this example, we could just write  $F = ma$ , but let us try the more general process.

$$\frac{\partial \mathcal{L}}{\partial x} = m\dot{x} \text{ and } \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx$$

And Hook's law tells us that  $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$ , or  $m\ddot{x} = -kx$  and  $\ddot{x} = -\frac{kx}{m}$ . The solution to this equation is cosines and sines. Why?

Suppose I have  $x = \cos \omega t$  (where  $\omega$  is the angular frequency), and I differentiate it once...  $\dot{x} = -\omega \sin \omega t$ . And if I differentiate it again, I get  $\ddot{x} = -\omega^2 \cos \omega t = -\omega^2 x$ . This is similar to  $\ddot{x} = -\frac{kx}{m}$ . It says that when I take the second derivative, I get back a function which is proportional to itself, but multiplied by a negative. So,  $\omega^2 = \frac{k}{m}$  or  $\omega = \sqrt{\frac{k}{m}}$ . This works for sin as well, as well as in the linear combination of sin or cosine.

The solution can be written either as a general solution with coefficients  $a$  and  $b$ ...

$x = a \cos(\omega t) + b \sin(\omega t)$  or as  $x = A \cos[\omega(t - t_0)]$  where  $A$  tells us how big the oscillations are (amplitude), and  $t_0$  tells you when the largest amplitude occurs (**the phase**). If you were to pull on a spring, and then let it go, the moment you let it go would be represented by  $t_0$ .

Now let us think about its Hamiltonian, and its canonical momenta.

$$P = \Pi_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$H = P\dot{x} - \mathcal{L} = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{k}{2}x^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ . Which is what we expected, and as we know this quantity (the energy) is conserved.

Up until now, we have worked with the Lagrangian formulation of the equations of motion, now we will turn to the Hamiltonian formulation.

Instead of working with  $q$  and  $\dot{q}$  or  $x$  and  $\dot{x}$ , The Hamiltonian works with the basic variables of canonical momenta and coordinates ( $qs$  and  $Ps$ ). Why are we doing this? it works well in quantum mechanics (which we won't be covering), and the formulations are very elegant. The goal is to write the equations not in terms of positions and velocities, but rather to write the equations in terms of positions and canonical momenta. As an aside, in quantum mechanics, there is a difference regarding the zero point energy. This is the understanding that at the quantum level, there is a minimum amount of energy necessary to disturb a particle. So, the accuracy that normally increases as the amount of oscillation decreases has a lower limit.

Let us rewrite the previous equation in terms of  $qs$  and  $Ps$ .

$$P = m\dot{x}, \quad \dot{x} = \frac{P}{m},$$

$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\left(\frac{P}{m}\right)^2 + \frac{1}{2}kx^2 = \frac{1}{2m}P^2 + \frac{k}{2}x^2$ . This creates a symmetry between the  $Ps$  and the  $xs$ .

Imagine a Cartesian coordinate plane with  $x$  (location) being on the horizontal axis, and  $P$  (momentum) being on the vertical axis (**Phase Space**). A starting position in the graph would consist of a location

and a momentum  $(x_0, P_0)$ . Because we are using the Hamiltonian formulation, we will witness a symmetry between the two axes that would not have occurred if using the Lagrangian velocity based equations. As the system moves in the phase space, the location and velocity change. Our task is to track the trajectory as it moves through phase space. One tool that we have is that we know that the energy is conserved ( $E = \frac{1}{2m}P^2 + \frac{k}{2}x^2$ ). The formulation of the equation indicates that the shape of this function is going to be an ellipse. If the coefficients were equal, then the shape would be a circle, and the radius would be a function of  $E$ . So, from our starting point, the function will move along an ellipse. We can discover the intercepts by first setting  $P$  and then  $x$  equal to zero and solving for the other variable.

$$E = \frac{1}{2m}(0)^2 + \frac{k}{2}x^2 \Rightarrow x = \pm \sqrt{\frac{2E}{k}},$$

$$E = \frac{1}{2m}P^2 + \frac{k}{2}(0)^2 \Rightarrow P = \pm \sqrt{2mE}.$$

How long does it take for the particle (phase point) to orbit the system (complete the ellipse around the origin)? That depends upon  $\omega$  (frequency). The larger the  $\omega$ , the more quickly the function completes the cycle. Now, project the phase point onto the  $x$ -axis (we ignore the  $P$ , and just look at the  $x$ -coordinate). In this case, the point oscillates. When the point is furthest out relative to the  $x$ -coordinate, it is also moving the slowest. The same relationship occurs when you project the phase point onto the  $P$ -axis. You can think of these two motions along the two axes as the projection of the circular phase space motion on to the  $x$ -axis and onto the  $P$ -axis.

Not all systems can be mapped onto an ellipses, but they will all map onto some contoured shape that maintains conservation of energy.

If you took a small piece of your phase space, you can consider this as being an area of uncertainty regarding where you began. You know that you began somewhere within that small piece of phase space, but that is all you know. As the function propagates, that small area of phase space also propagates with the function. We will come back to this idea...

## Lecture 6: Hamiltonian Formulation

Definition of a phase space: a system in which if you know the current state of the system, you know the subsequent state of the system, and the previous state of the system. This requires a system that has rules which make this predictive and retracing ability possible. In a phase space where the rules allow points to bifurcate either forward in time or backward in time, locations in the phase space can become un-occupyable. Hamiltonian space does not allow for this. This is also called information conservation.

Equations that are in the realm of  $F = ma$  type equations are not in a form that satisfies the above criteria. Why not? Let's imagine a three-dimensional space, and a particular point in that space for which you know the location ( $q$ ). In order to know where the particle will be next, we need to know not only where it is now, but where it was a moment ago. In other words, I need to know both the position and the velocity ( $\dot{q}$ ). So, phase space is a way of describing a system which gives you enough information to make these predictions.

The number of equations that define a system in three dimensions is equal to the number of particles in that system times three. For each particle, the three equations are for each of the three dimensions. Each of these equations is a second order differential equation (like  $F = ma$  where acceleration is the second derivative with respect to time). Hamilton's formulation allows you to take these second-order differential equations (let's say there are  $n$  of them) and replace them with twice as many first-order

differential equations. While the process is completely general, let us try it for equations of the form  $F = ma \dots$

There are many ways of doing this, but for today let us assume that every particle in our system has the same mass.

$$ma_i = m \frac{d^2x_i}{dt^2} = F_i \Rightarrow n \text{ of them}$$

Define  $m \frac{dx_i}{dt} = p_i$ .

Then,  $F_i = \frac{dp_i}{dt}$ . Therefore, for each particle there are two equations (both of first order), and now there are twice as many variables. In this case, instead of just one variable ( $x_i$ ), there are now two variables ( $x_i, p_i$ ). This in some ways is merely a bit of redefinition. However, embedded in it is a different mindset. It allows one to know where one is going by merely knowing where one is in the phase space. It requires that you know the position and the velocity for each degree of freedom in the system (which in phase space is included in knowing where you are).

You can think of it in the following way: Classical mechanics represents a **flow in the phase space**. wherever you are you know where to go next. So, if you populate this phase space by taking a lot of identical systems (particles) and starting them at a lot of different points in the phase space, and watching them as they evolve (propagate), that would define a flow in the phase space. The flow in the phase space is what Hamiltonian physics is all about.

**Legendre Transformation** is generic, but the variables used ( $v, p$ ) can be thought of in many contexts as velocity ( $v$ ) and momentum ( $p$ ). These variables are actually functions of each other. These functions are also bijective so that if you know the value of one of the functions, you know the value of the other.

Let's define  $\mathcal{Q}(v)$  such that  $\frac{d\mathcal{Q}}{dv} = p$ . Given  $\mathcal{Q}$ , we can generate asymmetrically related function  $H(p)$ . When we think about  $\mathcal{Q}$ , we will think about  $v$  as the independent variable. And similarly with  $H$  and  $p$ .  $\mathcal{Q}$  is a function of  $p$  because  $p$  is a function of  $v$ . Similarly,  $H$  is a function of  $v$ , because  $p$  is a function of  $v$ .  $\frac{d}{dp} H(p) = v$ .

If we solve ( $\frac{d\mathcal{Q}}{dv} = p = p(v)$ ) for  $\mathcal{Q}$ , we get  $\mathcal{Q}(v) = \int_0^v p(v)dv$ . So,  $\mathcal{Q}(v)$  becomes the area under the curve between the origin and  $v$ . Similarly, if we solve ( $\frac{dH}{dv} = v = v(p)$ ) for  $H$ , we get  $H(p) = \int_0^p v(p)dp$ . So,  $H(p)$  becomes the area "under the curve" between the origin and  $p$ . Of course, if we imagine these two integrations on the same Cartesian coordinate plane, one would be the area beneath the curve as we normally think of it, and the other would be the area to the left of the curve (between the curve and the vertical axis). So,  $H + \mathcal{Q} = pv$  and  $H = pv - \mathcal{Q}$ .

Let's say we are given  $\mathcal{Q}(v)$ , we then differentiate it to get  $\frac{d\mathcal{Q}}{dv} = p$ . We then construct  $H$  by multiplying  $pv - \mathcal{Q} = H$ . But, because  $\mathcal{Q}$  is a function of  $v$ , we can instead replace these references with  $p$ .

$$H(p) = p \cdot v(p) - \mathcal{Q}(v(p)).$$

We will now prove that the derivative of  $H(p)$  in terms of  $p$  is  $v$ . How much does  $H$  change when we change  $p$  by just a little amount?

$$\delta H = p \cdot \delta v + \delta p \cdot v - \frac{d\mathcal{Q}}{dv} \cdot \delta v = p \cdot \delta v + \delta p \cdot v - (p) \cdot \delta v = \delta p \cdot v.$$

Therefore,  $\frac{dH}{dp} = v$ .

A similar formulation occurs when there are multiple  $v$ s. However, because there are now multiple variables, it is a partial derivative:  $\frac{\partial \mathcal{Q}}{\partial v_i} = p_i$  and  $\frac{\partial H}{\partial p_i} = v_i$ .

$$H(p) = \sum p_i v_i - \mathcal{Q}(v(p)).$$

$$\begin{aligned}\delta H &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \frac{d\mathcal{Q}}{dv_i} \cdot \delta v_i \\ &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \sum p_i \cdot \delta v_i \\ &= \sum v_i \delta p_i. \\ \frac{\partial H}{\partial p_i} &= v_i\end{aligned}$$

The above relationship between  $H$  and  $\mathcal{Q}$  is identical to the relationship between the Hamiltonian and the Lagrangian formulations of the equations of motion. The only detail we haven't looked at is how the locations of particles the  $q$ s enter into the equations. However, they "just go along for the ride."

Now let us assume we have our velocities  $\dot{q}_i$ , and let's instead refer to them as  $v_i$ . Then our Lagrangian becomes...

$$\mathcal{Q}(q_i, v_i)$$

Also, remember that  $p_i = \frac{\partial \mathcal{Q}}{\partial \dot{q}_i} = \frac{\partial \mathcal{Q}}{\partial v_i}$ .

So,  $\frac{\partial \mathcal{Q}}{\partial v_i} = p_i$ . And here we start to see the relationship between the derivative of the Lagrangian with respect to velocity, and the momentum. It is now the case that both our velocities and our canonical momenta are functions of both each other and of the  $q$ s ( $p(v, q)$  and  $v(p, q)$ ).

Now we will define Hamiltonian...

$$H(p) = \sum p_i \dot{q}_i - \mathcal{Q}(v(p, q)) = \sum p_i v_i - \mathcal{Q}$$

Now, let us calculate how the Hamiltonian changes when we change the  $p$ s and the  $q$ s by a small amount ( $\delta p$  and  $\delta q$ ).

$$\begin{aligned}\delta H &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \frac{d\mathcal{Q}}{dq_i} \delta q_i - \frac{d\mathcal{Q}}{dv_i} \delta v_i \\ &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \frac{d\mathcal{Q}}{dq_i} \delta q_i - \sum p_i \cdot \delta v_i \\ &= \sum v_i \delta p_i - \frac{d\mathcal{Q}}{dq_i} \cdot \delta q_i.\end{aligned}$$

$$\frac{\partial H}{\partial p_i} = v_i = \dot{q}_i.$$

$$\frac{\partial H}{\partial q_i} = -\frac{d\mathcal{Q}}{dq_i} \delta q_i.$$

Let's think of how we can interpret this last equation. We know that...

$$\frac{d}{dt} \frac{\partial \mathcal{Q}}{\partial \dot{q}_i} = \frac{\partial \mathcal{Q}}{\partial q_i} \text{ and therefore, } \dot{p}_i = \frac{\partial \mathcal{Q}}{\partial q_i}.$$

So,  $\frac{\partial H}{\partial q_i} = -\dot{p}_i$  and  $\frac{\partial H}{\partial p_i} = \dot{q}_i$ , which are Hamilton's equations.

$$H = \sum p_i v_i - \mathcal{Q}(q_i, v_i)$$

$$\begin{aligned}\delta H &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \frac{d\mathcal{Q}}{dq_i} \delta q_i - \frac{d\mathcal{Q}}{dv_i} \delta v_i \\ &= \sum p_i \cdot \delta v_i + \delta p_i \cdot \sum v_i - \frac{d\mathcal{Q}}{dq_i} \delta q_i - \sum p_i \cdot \delta v_i \\ &= \sum v_i \delta p_i - \frac{d\mathcal{Q}}{dq_i} \cdot \delta q_i.\end{aligned}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i.$$

$$\frac{\partial H}{\partial q_i} = -\frac{d\mathcal{Q}}{dq_i} \delta q_i = -\frac{d}{dt} p_i = -\dot{p}_i.$$

So, the equations of motion get greatly simplified into first-order equations. You only need to know one function of  $p$ s and  $q$ s, and through differentiation you can discover the regular velocity as well as the canonical momentum and therefore know the history of the system.

## Let's try it for $F = ma$ .

Let's make sure it works. Again, imagine a Cartesian coordinate with  $p$  ( ) being on the horizontal axis,

and  $q$  ( $\dot{q}$ ) being on the vertical axis. At each point in the plane, we can differentiate the Hamiltonian with respect to  $p$  and  $q$ , and calculate  $\dot{p}$  and  $\dot{q}$ . Doing so defines a velocity vector through the phase space. This is not a normal velocity, but rather a phase space velocity. If you were to do this with points throughout the phase space, and draw the vectors, you would see the flow through the phase space.

As an aside: If your system does not have a one-to-one relationship between your  $p$ s and  $q$ s (you cannot invert them, they lose their symmetry), it means the system will break down at some point into and infinity (for example, infinite velocity). It means that the system at some point does not know how a point should propagate (the rules are not clear).

Our new equations of mechanics ( $\frac{\partial H}{\partial p_i} = \dot{q}_i$  and  $-\frac{\partial H}{\partial q_i} = \dot{p}_i$ ) give us a phase space, a flow through the phase space, and a new way of thinking about the laws of mechanics. Let's see if our familiar equations work out with this new formulation...

$$\frac{1}{2}mv^2 - U(x) \Rightarrow H = \frac{p^2}{2m} + U(x)$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \rightarrow \text{It checks out (the relationship between velocity and momentum).}$$

$$-\frac{\partial H}{\partial q_i} \text{ should equal } \dot{p}_i, -\frac{\partial H}{\partial q_i} = -\frac{\partial U}{\partial x} = \dot{p}, \text{ (or } F = m \ddot{x} \text{ or } F = ma).$$

## Energy Conservation.

Can we prove that energy is conserved? Can we prove  $\frac{\partial H}{\partial t} = 0$ ?

$$\frac{\partial H}{\partial t} = \sum_i \left( \frac{\partial H}{\partial p_i} (-\dot{p}_i) + \frac{\partial H}{\partial q_i} \dot{q}_i \right)$$

Now let's use Hamilton's equations...

$$= \sum_i \left( -\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} \right) = \sum_i (0) = 0.$$

If we imagine the phase space as a topological structure, the previous equation means that the particle would travel along the topological lines equating to a constant energy level, as do all particles as they flow along the phase space flow. As with a topological map, these lines can also take a shape which is circular or elliptical. Our previous example in which we examined an elliptical structure within the phase space ( $H = p^2 + x^2$ ) is an example of this phenomena. The shape can also be merely a dot. This represents a system which is not changing. In the harmonic oscillator, this might relate to a node. In this topological view of phase space, the flow takes on a similar property as that found in thermodynamics. That is, as the lines of the topological map get closer together, the speed of the flow increases (like water flowing through a narrowing pipe). In the case of phase space, the increased flow relates to the Hamiltonian (or amount of energy) changing rapidly when the  $p$ s and  $q$ s are changing.

Now, let us not worry about why there is a conservation law, but rather what it means to have one.

Let us ask ourselves if  $A$  (a function of the phase space) is conserved. Wherever you are in the phase space,  $A(p, q)$  has a value. For example, we had previously worked out that the angular momentum of a particle moving in a plane is given by...

$xp_y - yp_x$ . In general, everything we can want to know about a particle is a function of its position and velocity. So, if there is a property we wish to discover about particle, there is a function which will tell us this property that consists of position and velocity as inputs to that function. What mathematical conditions (written in terms of Hamilton's equations) will cause  $A(p, q)$  to be conserved? This will introduce to us a new set of quantities, **Poisson brackets**. We wish to know if the time derivative of  $A$  is zero or not.

$$\frac{d}{dt}A(p, q) = \sum_i \left( \frac{\partial A}{\partial p_i} \dot{p}_i + \frac{\partial A}{\partial q_i} \dot{q}_i \right) = \sum_i \left( -\frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} \right) = 0.$$

More generally, take any two functions of  $p$  and  $q$ , ...

$A(p, q)$ ,  $B(p, q)$ , include them within the **Poisson brackets**  $\{A(p, q), B(p, q)\}$ , and their defined as...

$$\{A(p, q), B(p, q)\} = \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

Relating this to our previous example where we took the time derivative of  $A(p, q) = \dot{A}$ , this tells us that we can notate such a time derivative as  $\dot{A} = \{A, H\}$ . This is yet another formulation of the equations of mechanics.

Let's check it with an example.

$\dot{q}$  must be = to  $\{q, H\} = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = (1) \frac{\partial H}{\partial p} - (0) \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}$ . Or,  $\dot{q} = \frac{\partial H}{\partial p}$ , which is just Hamilton's equation.

$\dot{p}$  must be = to  $\{p, H\} = \frac{\partial p}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial p}{\partial p} \frac{\partial H}{\partial q} = (0) \frac{\partial H}{\partial p} - (1) \frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q}$ . Or,  $\dot{p} = -\frac{\partial H}{\partial q}$ , which is just Hamilton's equation.

So, Hamilton's equations are special cases of a much more general rule that the time derivative of any function is the Poisson brackets of the function with the Hamiltonian. Hamiltonians, therefore, produce time dependence by the action of Poisson brackets. The action of taking the Poisson brackets with the Hamiltonian tells you how systems behaves over time.

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Question from the audience: What would happen if the Hamiltonian were dependent upon time.

If we imagine the system we are studying to be a particle floating in space. And, instead of the normal situation where the potential energy is a function of  $x$ , imagine it were instead of function of both location  $x$  (which might implicitly depend on time) and explicitly time,  $U(x, t)$ . In the situation, when we take the Poisson brackets of this function, we get ...

$\frac{d}{dt} H(p, q, t) = \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} + \frac{\partial H}{\partial t} \right)$  where  $\frac{\partial H}{\partial t} = \frac{\partial U}{\partial t}$ . And since this time dependence implies that the value of this term is not zero, in general you will not get a zero value from this equation.

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## Lecture 7: Liouville's Theorem

Liouville's theorem is at the heart of classical mechanics, and a companion theory (unitary) is at the heart of quantum mechanics.

In phase space, particles existing in the phase space propagate in ways which are neither convergent (two points in the phase space flow into a common third point) nor divergent (a point in the phase space flows into two separate points). As a result, the phase volume is noncompressible. Another way of saying this is that, if the space begins with each particle in a separate part of the phase space, as the phase volume propagates, at no time will there be particles in the same part of the phase space.

Now imagine a bunch of particles in the phase space. As these particles propagate, because the phase volume is noncompressible, the area/volume the phase volume occupies maintains a constant volume. Equivalently, if you are to measure the vacant space around an individual particle, as the system propagates, the volume of vacant space around the particle remains constant. The phase volume will also maintain connectedness. This is a result of the fact that the rules of the system tell each particle where to go within the phase volume. So, a particle going off on its own would represent the system failing to tell the particle the proper place to go within the phase volume. The distance between the points, however, is not maintained. Volume is conserved, but not linear distance. In phase space, we define chaotic systems as systems which evolve in such a way as to become radically different in shape, as opposed to systems which maintain their original shape.

We are going to prove Liouville's Theorem by using Hamilton's equations.

The flow of the particles in a phase space move in a characteristic way governed by Hamilton's equations. The flow is determined by a single function whose input is all of the  $p$ s and  $q$ s. This means, if you know the Hamiltonian equation, and you are given a starting point for a system, you can follow that system through time determining its state. At any point in the phase space, a given particle will have two coordinates:  $(\dot{p}, \dot{q})$ . The time derivatives of the  $p$ s, or  $\dot{p}$  relates to the velocity of the particle in the direction  $p$ , and similarly with the time derivative of the  $q$ s (the regular velocities in normal space), or  $\dot{q}$ .

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Let us show what it means for a flow to be incompressible. We are going to start with a one-dimensional flow. Imagine a line with particles spread out along the line in a uniform fashion. Incompressible flow in this context would be observed if the points were all moving in tandem (the same velocity). If the points to the left on this line traveled to the right faster than the points on the right traveled, we would see a clumping of points, an increase in their density. There are two ways of looking at this, one view is to take a look at a few of the particles. The range (volume) these particles occupy must remain constant through time for the system to be considered incompressible. Another view is to choose a range within the phase volume and note the number of particles within that range. As time passes, if the phase volume is incompressible, the number of particles which leave that range as time passes is always equal to the number of particles which enter it. So the density of that arbitrary range is conserved. So, in this arbitrary range, if particles have a higher velocity ( $v_2$ ) and are therefore leaving more quickly at one end of the range than are entering at the other end ( $v_1$ ), this does not represent an incompressible phase volume. In this case, the density is changing by  $(v_2 - v_1) \cdot (\text{The density of the points})$ . However, if the density is assumed to be distributed evenly, then this reduces to  $(v_2 - v_1)$ . And, if we are deciding that the phase volume is incompressible, this must mean that  $v_2 - v_1 = 0$ . For increasingly small intervals, the same idea can be expressed as  $\frac{\partial v}{\partial x} \Delta x$  where  $\Delta x$  is the interval. So,  $\frac{\partial v}{\partial x} \Delta x = 0$ . And, since  $\Delta x$  is just a number and not a variable, we can reduce this to  $\frac{\partial v}{\partial x} = 0$ .

Two dimensional phase space ( $x$  and  $y$ ): Imagine a rectangular phase volume whose size is  $\Delta x \cdot \Delta y$ , and whose orientation aligns the sides of the rectangle with the coordinate axes. Let us label the sides of the rectangle  $r_1, r_2, r_3$ , and  $r_4$ , with  $r_1$  being the side of the rectangle whiches closest to the Y axis and parallel to it,  $r_2$  being the top of the rectangle, and so on clockwise. How many points are entering and leaving each side of this rectangle? We imagine that initially the density across the phase space is uniform. The number of particles entering the rectangle through  $r_1$  per unit time must be proportional to the X component of velocity, and the Y component must not contribute anything.  $v_x \cdot \Delta y$  therefore represents the particles entering  $r_1$ . However, a similar but opposite phenomena occurs on  $r_3$  such that  $-v_x \cdot \Delta y$  represents the particles leaving the space. However, the velocity is not the same at  $r_1$  and  $r_3$ . The difference between these velocities is obtained with  $\frac{\partial v_x}{\partial x} \Delta x$ . Therefore, the change in particles between these two sides can be represented as  $\frac{\partial v_x}{\partial x} \Delta x \Delta y$ . For the horizontal edges, we have a similar situation where the net number of particles coming into the system is represented as  $\frac{\partial v_y}{\partial y} \Delta y \Delta x$ . So, the total number of points entering the system is equal to  $\frac{\partial v_x}{\partial x} \Delta x \Delta y + \frac{\partial v_y}{\partial y} \Delta y \Delta x = \Delta x \Delta y \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right]$ .

$\left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right]$  or  $\nabla \cdot v$  is referred to as the divergence of the velocity field (or more simply, the divergence). If it is positive, it accompanies an increase in the density, and a decrease when it is negative. However, if this is an incompressible fluid, this quantity must be equal to zero,  $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ . Note that this equation does not require the velocity to remain constant from place to place, but rather if it is



increasing along the x-coordinate, it must simultaneously be decreasing along the y-coordinate. More generally with any number of dimensions,  $\sum_i \frac{\partial v_i}{\partial x_i} = 0$  or  $\nabla \cdot \mathbf{v} = 0$ . You can make a similar equation while not assuming the uniformity of density, but this results in the adding of a density factor.

Now let us apply this concept to phase space. Instead of  $x_s$ , we have  $p_s, q_s$ . However, for reasons we mentioned earlier, there are twice as many terms; for each  $q$  (for a spatial coordinate, for example), we also have a  $p$ . In this scenario,  $\dot{p}$  and  $\dot{q}$  are the local velocities or the  $v_i$ s. So,  $\dot{p} = v_{p_i}$  and  $\dot{q} = v_{q_i}$ .

Now let us calculate the divergence of the flow. We will use Hamilton's equations to see that the divergence of the flow is exactly 0. In one dimension,  $\frac{\partial v_p}{\partial p} = \frac{\partial}{\partial p} \left( -\frac{\partial H}{\partial q} \right)$  and  $\frac{\partial v_q}{\partial q} = \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} \right)$ . So,  $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial p} \left( -\frac{\partial H}{\partial q} \right) + \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} \right)$ .

And, if we had multiple dimensions, this would change to...

$$\nabla \cdot \mathbf{v} = \sum_i \left[ \frac{\partial}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) + \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) \right].$$

However, because of the nature of partial differentiation we know that  $\frac{\partial}{\partial p} \left( \frac{\partial H}{\partial q} \right) = \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} \right)$ . So,  $\frac{\partial}{\partial p} \left( -\frac{\partial H}{\partial q} \right) + \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} \right) = 0$ . Therefore, the flow in phase space is incompressible. This is a profound fact about Hamiltonian mechanics. Remember, there is no conservation of distance in the phase space, only volume.

Let's take a two-dimensional phase space example...

We are going to simplify this by setting the mass  $m = 1$ .

So, the kinetic energy of a particle moving along the X axis is  $\mathcal{L} = \frac{x^2}{2}$ .

$$p_x = \frac{d}{dt} \frac{x^2}{2} = \dot{x}.$$

Now, let's try the same thing but with a new coordinate we will call  $y$  (not necessarily the vertical axis you are used to, but actually in the same direction as the previous axis but with a different quality...). In this case  $y = \alpha x$ , where  $\alpha$  is an arbitrary constant (this end up being a unit transformation).

$$\frac{y}{\alpha} = x \text{ and } \dot{\frac{y}{\alpha}} = \dot{x}$$

Let's rewrite the Lagrangian. Its value should not change when we change coordinates...

$$\mathcal{L} = \frac{\left(\frac{\dot{y}}{\alpha}\right)^2}{2} = \frac{\dot{y}^2}{2\alpha^2} \text{ and } p_y = \frac{d}{dt} \frac{\dot{y}^2}{2\alpha^2} = \frac{\dot{y}}{\alpha^2} = \frac{\dot{x}}{\alpha} = \frac{p_x}{\alpha}.$$

So,  $y = \alpha x$ ,  $p_y = \frac{1}{\alpha} p_x$ . Notice that if you stretch the  $x$ -axis, you shrink the  $p$ -axis.

Liouville's Theorem is the statement that the volume in phase space is incompressible.

Area has an invariant meaning in phase space, and becomes the concept of uncertainty in quantum mechanics.

In extremely complicated systems (chaotic systems), to practically model these systems, it becomes necessary to do something other than follow every individual particle. Rather, one instead defines an amount of volume large enough that one can divide up the initial system into sectors that have that amount of volume, and that one can then follow the phase space while maintaining an accuracy down to that amount of volume. The difficulty with this method (course grained), is that if you then add up all of the volumes at the end of the timeframe in which you are watching the system, the total volume will have changed. In reality, this has not occurred, and is therefore a pitfall to following a system in this manner. This is the origin of the second law of thermodynamics. The entropy of a system is the detectable volume of the phase space that the system occupies. If you know that a system exists

somewhere within a particular volume of space space, Entropy is the logarithmic measure of that volume. And, because you haven't be precision to follow the system carefully enough, the detectable volume of the phase space which includes to propogated system necessarily increases. That's why entropy increases. It's a statement that when a phase space propogates you lose information, not because information is inherently lost in the universe, but because you have a limitation to the degree you can follow the information.

And now for something completely different...

Let's work out how will particle moves in an electromagnetic field.

The new thing in this type of the situation is velocity dependent forces. So far, the forces we have encountered depend upon where you are, but not how you are moving.

In our previous examples, we differentiated the potential energy with respect to  $x$ , to discover the force (which was a function of position  $x$ ):  $-\partial_x U(x) = F(x)$ . However, there are forces in nature that depend upon your velocity (a magnetic field acting on a particle is a prime example). Friction is also an example. If an object is not moving, there is no friction, and the faster the object is moving the more friction occurs. The difference between a magnetic field force and a friction force includes the fact that the magnetic force is derivable from the principle of least action, has a Lagrangian formulation, there is a conservation of energy, and there is a Hamiltonian formulation. Although, there is a new twist to the Lagrangian and Hamiltonian formulation, which we will now work out...

Let's start out with the force that a magnetic field has on a charged particle.

$\vec{B}(x)$  is a vector representing the magnetic field, and depends upon the location of the particle.

The force on a particle is equal to  $q[\vec{v} \times \vec{B}]$  where  $q$  is the charge of the particle, and  $v$  is the velocity of the particle.

$$F_x = \left[ \vec{v} \times \vec{B} \right]_x = v_y B_z - v_z B_y, \quad F_y = \left[ \vec{v} \times \vec{B} \right]_y = v_z B_x - v_x B_z, \quad F_z = \left[ \vec{v} \times \vec{B} \right]_z = v_x B_y - v_y B_x.$$

$$\vec{F} = \left[ \vec{v} \times \vec{B} \right] = \{v_y B_z - v_z B_y, v_z B_x - v_x B_z, v_x B_y - v_y B_x\}.$$

New concept: Vector Potential ( $\vec{A}$ ) (a way of notating magnetic fields)

It is needed for writing the equations in the form of a Lagrangian or a Hamiltonian formulation.

The magnetic field is the curl of the vector potential.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Vectors which are curls of other vectors has no divergence. A magnetic field has no divergence, which is why it is convenient to write it this way.

$$\vec{B}_x = \left[ \vec{\nabla} \times \vec{A} \right]_x = \partial_y A_z - \partial_z A_y, \quad \vec{B}_y = \left[ \vec{\nabla} \times \vec{A} \right]_y = \partial_z A_x - \partial_x A_z, \quad \vec{B}_z = \left[ \vec{\nabla} \times \vec{A} \right]_z = \partial_x A_y - \partial_y A_x.$$

$$\vec{B} = \left[ \vec{\nabla} \times \vec{A} \right] = \{\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x\}.$$

So, the force on a charged particle due to a magnetic field...

$$\vec{F} = q \left[ \vec{v} \times \vec{B} \right] = q \left[ \vec{v} \times \left( \vec{\nabla} \times \vec{A} \right) \right].$$

$$F_z = q[v_x B_y - v_y B_x] = q[v_x(\partial_z A_x - \partial_x A_z) - v_y(\partial_y A_z - \partial_z A_y)],$$

$$F_x = q[v_y B_z - v_z B_y] = q[v_y(\partial_x A_y - \partial_y A_x) - v_z(\partial_z A_x - \partial_x A_z)],$$

$$F_y = q[v_z B_x - v_x B_z] = q[v_z(\partial_y A_z - \partial_z A_y) - v_x(\partial_x A_y - \partial_y A_x)].$$

$F = F_x + F_y + F_z$ . However, now we see that these terms of the force depend not only on position ( $A$ ), but also on velocity ( $v$ ).

We want to formulate equations of motions for a charged particle moving in a magnetic field using the Lagrangian or the Hamiltonian form. It is not immediately obvious that we can do this. The easiest way to do this is to really make a guess as to as to what the action is, and then to follow it through. We do

expect that if the charge of the particle is zero, that the normal parts of the equation will occur. Our guess...

$\int \mathcal{L} dt = \int \frac{mv^2}{2} dt$  where  $v^2$  is the sums of the squares of the x-component, y-component, and z-component of the velocity.

Then we add another term. Imagine a charged particle moving through its orbit, break the path up into little segments. Each segment we could label  $dx$ ,  $dy$ , and  $dz$ ; but for simplicity, let us just label it as  $\vec{dx}$ . The added term is proportional to the electric charge  $q$ , this allows the term to disappear if there is no magnetic field or charge.

$$\int \mathcal{L} dt = \int \frac{mv^2}{2} dt + q \int \vec{A} \cdot \vec{dx}$$

What this last term means, is that at every point in space you have a vector potential ( $\vec{A}$ ), as the particle sweeps out along its trajectory, there is a contribution to its action along each  $\vec{dx}$ . This contribution is calculated by taking the dot product of the action potential ( $\vec{A}$ ) with the amount of displacement  $\vec{dx}$ . Notice that this equation does not have the normal form of an action which depends upon velocities and positions. The vector potential depends upon positions normally, but  $\vec{dx}$  is a problem. To fix it, however, all we need to do is multiply by  $\frac{dt}{dt}$ ...

$$\begin{aligned} \int \mathcal{L} dt &= \int \frac{mv^2}{2} dt + q \int \left( \vec{A} \cdot \vec{dx} \frac{dt}{dt} \right) \\ &= \int \frac{mv^2}{2} dt + q \int \left( \vec{A} \cdot \frac{d\vec{x}}{dt} \right) dt \\ &= \int \frac{mv^2}{2} dt + q \int \left( \vec{A} \cdot \vec{v} \right) dt \\ &= \left\{ \int \sum_i \left[ \frac{m\dot{x}_i^2}{2} + q(A_i \dot{x}_i) \right] dt \right\}. \end{aligned}$$

This is now in our usual form. Our Lagrangian can now be written as depending upon position and velocity, as it should.

$$\mathcal{L} = \sum_i \left[ \frac{1}{2} m \dot{x}_i^2 + q(\dot{x}_i A_i) \right].$$

The next step is to prove that the equations of motion for the situation are equal to  $F = ma$ , where  $ma = q(\vec{V} \times \vec{B})$ .

First, what is the canonical momenta  $p_x$ ? It is the integral of the Lagrangian with respect to  $\dot{x}$ .

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} + qA_x, \quad p_y = m \dot{y} + qA_y, \quad p_z = m \dot{z} + qA_z.$$

Notice that the canonical momenta now has an additional term that depends upon the charge of the particle and the magnetic field. Sometimes  $m \dot{x}$  is referred to as the mechanical momentum, and the whole value is referred to as the canonical momentum. Let's work out the z-component of the equations of motion, which is the time derivative of the momentum ( $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{d}{dt} p_z = \frac{\partial \mathcal{L}}{\partial z}$ ), .

$$p_z = m \dot{z} + qA_z.$$

$$\frac{d}{dt} p_z = m \ddot{z} + q \dot{A}_z$$

That was the left-hand of our equation ( $\frac{d}{dt} p_z = \frac{\partial \mathcal{L}}{\partial z}$ ), now let us work out the right-hand side...

$$\frac{\partial \mathcal{L}}{\partial z} = q \left[ \dot{x} \frac{\partial A_x}{\partial z} + \dot{y} \frac{\partial A_y}{\partial z} + \dot{z} \frac{\partial A_z}{\partial z} \right]$$

Now back to the left-hand side of our equation, how do we calculate  $\dot{A}_z$ ? It (the time derivative of the magnetic field) is not equal to zero. As the particle moves through the field (even though where using an example where the field itself does not change over time), the particle experiences time-dependent changes in the field as it moves through it. So,

$$\frac{d}{dt} p_z = m \ddot{z} + q \dot{A}_z = m \ddot{z} + q \left( \frac{\partial A_z}{\partial x} \dot{x} + \frac{\partial A_z}{\partial y} \dot{y} + \frac{\partial A_z}{\partial z} \dot{z} \right) = q \left[ \dot{x} \frac{\partial A_x}{\partial z} + \dot{y} \frac{\partial A_y}{\partial z} + \dot{z} \frac{\partial A_z}{\partial z} \right], \text{ (from above).}$$

By canceling we get...

$$m \ddot{z} + q \left( \frac{\partial A_z}{\partial x} \dot{x} + \frac{\partial A_z}{\partial y} \dot{y} \right) = q \left[ \dot{x} \frac{\partial A_x}{\partial z} + \dot{y} \frac{\partial A_y}{\partial z} \right]$$

$$m \ddot{z} = q \left[ \dot{x} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \dot{y} \left( \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \right) \right]$$

$$m a_z = q \left[ v_x \vec{B}_y - v_y \vec{B}_x \right] = q \left[ \vec{v} \times \vec{B} \right]_z, \text{ (the substitutions were retrieved from definitions listed above)}$$

As you can see, the Lagrangian depends upon the vector potential ( $A$ ), but the equation of motion only depends upon the magnetic field  $\vec{B}$ . So, even if there is an ambiguity in the vector potential, it doesn't affect the equation of motion itself.

So, we now have an example of a velocity dependent force. It is almost like a friction force, except the friction force acts along the direction of velocity. This is an important difference. When a velocity dependent force acts only along the direction of velocity, it has the limited effect of either speeding up or slowing down the particle. If, on the other hand, the velocity dependent force can act in other directions, it can have the effect of curving the trajectory, and it won't affect the energy (energy conservation).

Now let us work out the Hamiltonian, so that we can determine what is conserved. We are assuming that the magnetic field itself is unchanging overtime. This allows for conservation of energy. The conservation is easy to express in terms of velocities, but not as easy in terms of momenta.

$$p_x = m \dot{x} + qA_x, \quad p_y = m \dot{y} + qA_y, \quad p_z = m \dot{z} + qA_z.$$

Let's calculate, not in terms of  $p$  (momenta), but instead in terms of  $\dot{x}$  (velocities).

$$H(p) = \sum p_i v_i - \mathcal{L}, \text{ (from earlier in the course)}$$

$$H(p) = (p_x \dot{x} + p_y \dot{y} + p_z \dot{z}) - \mathcal{L}$$

This equation takes each momenta, multiplies it by the relevant velocity, and subtracts the Lagrangian. If I plug in for the  $p$ s, their expressions in terms of velocities, I'll have an expression for the energy in terms of velocities. That won't be good for Hamilton's equations, but it will be good for telling me what the energy is in terms of velocities. So, let's do it...

$$H(p) = \left[ (m \dot{x} + qA_x) \dot{x} + (m \dot{y} + qA_y) \dot{y} + (m \dot{z} + qA_z) \dot{z} \right] - \mathcal{L}$$

$$= (m \dot{x}_i + qA_{x_i}) \dot{x}_i - \left[ \frac{1}{2} m \dot{x}_i^2 + q(\dot{x}_i A_{x_i}) \right]$$

$$= m \dot{x}_i^2 + q \dot{x}_i A_{x_i} - \frac{1}{2} m \dot{x}_i^2 - q \dot{x}_i A_{x_i}$$

$$= \frac{1}{2} m \dot{x}_i^2, \text{ Which is exactly the energy we get when we calculated the initial kinetic energy. The}$$

magnetic field does not contribute to the energy when expressed in terms of velocity. Why is that? That is equivalent to the statement that magnetic fields do no work. It's because the force is perpendicular to the velocity. Because the force on a particle moving in a magnetic field is perpendicular to the velocity, the magnitude of the velocity does not change. Its direction may change, but not its speed. But this form does not let us write the equation in terms of a Hamiltonian. To do this, we must express it in terms of momenta ( $p$ ).

$$\dot{x} = \frac{p_x - qA_x}{m}$$

$$H_x = \frac{1}{2} m \left( \frac{p_x - qA_x}{m} \right)^2 = \frac{(p_x - qA_x)^2}{2m}, \quad H_y = \frac{(p_y - qA_y)^2}{2m}, \quad H_z = \frac{(p_z - qA_z)^2}{2m}$$

$$H = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m} = \frac{1}{2m} \left[ (p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2 \right].$$

Exercise: Take this Hamiltonian, workout Hamilton's equations, and check that you get the same equations of motion.  $m \vec{a} = (\vec{v} \times \vec{B})$

## Lecture 8

Let's do more with a charged particle in an electromagnetic field. We're not going to worry about (in this course) the dynamic nature of the field itself. For our purposes, the electric and magnetic fields are merely given to us. For simplicity, we are going to assume that these fields do not depend on time (they are static), although they may depend on space. First, let us rework out the situation when a particle is in a magnetic field, as the situation is not intuitive.

Let's make a list of the various forms of mechanics we've covered so far:

- Equations of motion (e.g.,  $F = ma$ ) which are differential equations of second order.
- Principle of Least Action
- Lagrange's equations of motion (derived from the principle of least action by applying the Euler-Lagrange equations)
- Hamiltonian Form (by interpreting Lagrange in a phase space. The advantage is the ability to observe a flow through phase-space and information conservation, the disadvantage is that it is less intuitive to calculate)

Equation for a non-relativistic particle (slow-moving) moving through an electromagnetic field:

$F = m\vec{a} = q\vec{E} + q(\vec{v} \times \vec{B})$  where  $q$  is the charge of the particle,  $\vec{E}$  is the electric field. In some usages, the second term includes the speed of light in the denominator. However, for our usages we will absorb that constant into another quantity (either into  $q$ , into the magnetic field, or into your definition of  $\vec{v}$ ). And as we learned before, the force of magnetism on the particle is not along the direction of the velocity, and therefore neither adds nor subtracts from the speed. It therefore does not affect the overall kinetic energy of the system (magnetic fields do no work). This can be observed by the fact that the magnetic force is a cross product of the vector, and cross products are always at right angles to the vectors involved.

Imagine the field through which a particle will be traveling as a two-dimensional plane (the whiteboard) with the magnetic field oriented perpendicular to the whiteboard, and the electric field oriented parallel to the whiteboard. The cross product of the magnetic field is also therefore parallel to the whiteboard, and therefore all alterations of the trajectory due to magnetism also occur parallel to the whiteboard (allowing us to visualize it on this plane). To make the step from a simple equation of motion (listed above) to one of the more useful forms of mechanics we've studied, you need the idea of the vector potential ( $\vec{A}$ ).

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Vectors which are curls of other vectors has no divergence. A divergence is the spread of a vector field away from a point. We know that a magnetic field has no divergence because it can be written in the form of a curl of another vector. Although, there may be many ways to write such a curl (there is not one unique way to write it, which becomes important). This means that you can change the vector potential you use without changing the magnetic field, and therefore without changing the equations of motion (this is called gauge invariance).

The electric field is a more ordinary, non-velocity dependent force. It depends upon position, and it is conservative (in that it respects the conservation of energy). And we know that conservative forces are generated from potential energy functions. Force is equal to minus the gradient of a potential energy function ( $F = -\nabla U$  or  $F_i = -\frac{\partial}{\partial x_i} U$ ). For energy, we write...

$\vec{E} = -\nabla v$  where  $v$  (measured in volts, which is an energy per unit charge,  $\frac{\text{energy}}{\text{unit charge}}$ ) is like potential energy, but it is missing the charge. So, potential energy of a charge is  $U = qv$ . This tells us that the

potential energy of a charge is proportional to its charge (if you double the charge, you double the potential energy).

Now let us discuss the principle of least action. The only known formula for an action which gives rise to the equation ( $F = q\vec{E} + q(\vec{v} \times \vec{B})$ ), requires you to use the vector potential. You cannot avoid making the substitution. This is rather odd because the behavior of the particle does not depend upon the non-unique vector potential, but rather upon the magnetic field. Let's compute the action of the particle...

$$\text{Action} = \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - q \cdot v(x,y) \right] dt + \int \left[ q (\vec{A}_i \cdot \vec{dx}_i) \right]$$

The second term is of course the contribution from the magnetic field.

Let's move one of the terms, and look at it a different way...

$$\text{Action} = \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right] dt + \int \left[ q (\vec{A}_i \cdot \vec{dx}_i) - qv dt \right] = \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right] dt + q \int \left[ (\vec{A}_i \cdot \vec{dx}_i) - v dt \right]$$

So, for the second term we have the vector potential in each direction being multiplied by a small change in that direction, and for the time component, we have  $v dt$ . It's as if we plotted the motion in space time, we are not talking about relativity for the moment, but you can see that what is being set up here is going to be particularly simple when we think about the special theory of relativity, where  $x$  and  $y$  form space-time. So, imagine a three-dimensional Cartesian set of coordinates in which  $x$ ,  $y$  and  $t$  form the axes. And imagine a particle moving through this space. Now break up the trajectory into little pieces. Now, turning back toward equation, it's as if  $v$  is the fourth component of a vector potential (including all three directions of space and time, dotted with a fourth dimension of voltage).

The  $\vec{A}_i \cdot \vec{dx}_i$  should look a little unfamiliar. Up until now we have been integrating a Lagrangian, that has been a function of velocities and positions.  $\vec{A}_i$  and  $v$  are functions of position. However, there is this odd form of  $\vec{A}_i$  being multiplied by  $\vec{dx}_i$ . However, we can rewrite it in a more familiar way by multiplying by  $\frac{dt}{dt}$ ...

$$\begin{aligned} \text{Action} &= \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right] dt + q \int \left[ (\vec{A}_i \cdot \vec{dx}_i \cdot \frac{dt}{dt}) - v dt \right] \\ &= \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right] dt + q \int \left[ (\vec{A}_i \cdot \frac{\vec{dx}_i}{dt}) dt - v dt \right] \\ &= \int \left[ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right] dt + q \int \left[ \vec{A}_i \cdot \vec{v}_i - v \right] dt, \text{ where } \vec{v}_i \text{ is the velocity, and } v \text{ is the measurement and volts.} \end{aligned}$$

And now to give the Lagrangian, we just remove the integration signs...

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + q [\vec{A}_i \cdot \vec{v}_i - v]$$

**Exercise:** And the construction of the Hamiltonian would proceed similarly to how we did it in the last lecture.

Now let us focus on the concept of a gauge invariance...

What kind of things can you do to the vector potential that do not change the magnetic field? It does not change the underlying physics, and therefore there must be some redundancy in the description. We can also think of it as a symmetry of the system where changing the vector potential in certain a way, has no influence on the motion. To help us understand it, let's focus on a particular component of

$$\left( \vec{\nabla} \times \vec{A} \right)_z$$

$$\left( \vec{\nabla} \times \vec{A} \right)_z = \partial_x A_y - \partial_y A_x$$

Suppose that I add to the vector potential, something which is itself a gradient.

$\vec{A} + \nabla\lambda(x,y) = \sum_i [A_i + \frac{\partial}{\partial x_i} \lambda(x,y)]$  where  $\lambda$  can be any function. The point is, changing the vector potential in this way does not change the curl of the vector potential. Let's see why this is.

We are going to recalculate the curl of  $\vec{A}$ , but with our addition.

$$A_x + \frac{\partial}{\partial x} \lambda(x,y), \quad A_y + \frac{\partial}{\partial y} \lambda(x,y),$$

$(\vec{\nabla} \times \vec{A})_z = (\partial_x A_y - \partial_y A_x) + (\partial_x \partial_y \lambda - \partial_y \partial_x \lambda) = (\partial_x A_y - \partial_y A_x)$ . So, by adding the gradient of a scalar, it does not change the curl of a vector. The gradient of a scalar does not have a curl.

Let's work out the equations of motion, starting with the Lagrangian and see that it really does have the form..( $F = m\vec{a} = q\vec{E} + q(\vec{v} \times \vec{B})$ ). As an aside, the form of this equation  $F = q\vec{E} + q(\vec{v} \times \vec{B})$  is referred to as the **Lorentz force law**. Lets just do this for the X component. We first calculate the canonical momentum conjugate to  $x$ . That's the derivative of the Lagrangian with respect to  $\dot{x}$ .

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q[\vec{A}_i \cdot \vec{v}_i - v]$$

$p_x = m\dot{x} + qA_x$ , (the second term comes from the fact that  $q\vec{A}_i \cdot \vec{v}_i$  contains within it  $qA_x v_x = qA_x \dot{x}$ ). So, we end up with an extra term that is dependent upon the vector potential. This is strange because the new term is not gauge invariant (in that it depends upon the vector potential). The canonical momentum of a point particle within a magnetic field is not gauge invariant. Later, we will give examples of having different vector potentials to see how this works. Let's see the equation of motion which is the time derivative of this canonical momentum.

$$\frac{d}{dt} p_x = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

$\frac{d}{dt} p_x = m\ddot{x} + q(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y})$ , (While the magnetic field does not change with time, the force upon the particle as the particle moves along its trajectory was time does change. Therefore, the second term of this equation when differentiating with time does not disappear).

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x} q(A_x \dot{x} + A_y \dot{y} - v(x,y)) = q(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_y}{\partial x} \dot{y} - \frac{\partial v}{\partial x})$$

$$\text{So, } m\ddot{x} + q(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y}) = q(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_y}{\partial x} \dot{y} - \frac{\partial v}{\partial x})$$

$$\Rightarrow m\ddot{x} + q(\frac{\partial A_x}{\partial y} \dot{y}) = q(\frac{\partial A_y}{\partial x} \dot{y} - \frac{\partial v}{\partial x})$$

$$\Rightarrow m\ddot{x} = q[(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \dot{y} - \frac{\partial v}{\partial x}]$$

$$\Rightarrow m\ddot{x} = q[B_z \dot{y} - \frac{\partial v}{\partial x}], B_z \dot{y} \text{ is exactly the same as the x-component of } \vec{v} \times \vec{B}$$

$$([\vec{v} \times \vec{B}]_x = v_y B_z - v_z B_y, \text{ from above}).$$

Also,  $-\frac{\partial v}{\partial x}$  ends up equaling  $\vec{E}$ , since  $F_i = \frac{\partial}{\partial x_i} U$ ,  $\vec{E} = -\nabla v$ ,  $F = -\nabla U$  and  $U = qv$  (from above).

$$-\frac{\partial v}{\partial x} = -\frac{1}{q} F_x = -\frac{1}{q} (-\nabla U_x) = \frac{1}{q} \nabla(qv) = \nabla v = -\vec{E}_x.$$

$$F_x = m\ddot{x} = q\vec{E}_x + q(\vec{v} \times \vec{B}_x)$$

Again, even though we had to use the vector potential to complete our calculations, when we arrived at our equations of motion, only the term for the magnetic field appeared in our equations. So, the vector potential appears in a gauge invariant way. Often in physics, in order to have your equations notated in Lagrangian, least action, or Hamiltonian form, you need to introduce redundant descriptions of things. We call these redundant descriptions of things **Gauge Invariances**. The transformations that correspond to these redundancies (the changes that you make that are really not changes at all in the underlying physics), those are called **Gauge Transformations**.

Now we're going to work with some examples of vector potentials, and in particular we are going to concentrate on the problem of a uniform magnetic field; one that does not vary from place to place. In our example, the magnetic field will be pointing into the blackboard. We're going to construct one from

a vector potential. We want  $B_z$  to be independent of position.

$$B_z = \partial_x A_y - \partial_y A_x$$

Here are two examples of vector potentials that satisfy this requirement: The magnetic field goes into the blackboard (meaning it is positive), and its magnitude is...

$$|B| = b$$

Make  $A_y = bx$ , and  $A_x = 0$ . In this case, the derivative of the  $A_y$  with respect to  $x$  is  $b$ .  $B_z$  therefore is equal to  $b$ . So, this is one set of vector potentials that will give rise to the unified magnetic field.

Another one is  $A_y = 0$  and  $A_x = -by$ . And another one is  $A_y = \frac{b}{2}x$  and  $A_x = -\frac{b}{2}y$ . These different vector potentials are related by gauge transformations. It's not too difficult to work out that there is some scalar that takes you from one vector potential to another by taking its gradient.

Let's see what we learn about these types of vector potentials from conservation laws. If we consider the gauge  $A_x = by$ , and  $A_y = 0$  (Gauge 1), the Lagrangian is (for simplicity we are going to ignore the electric field for the moment):

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q[\vec{A}_i \cdot \vec{v}_i] = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q[A_x \dot{x}], \text{ (because } A_y = 0\text{).}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q(by) \dot{x}, \text{ Now let's calculate the canonical momenta conjugate to } x \text{ and } y.$$

$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} + qby$ ,  $p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m \dot{y}$ , Which one of these is conserved?  $p_x$  (the x-component of momentum) is conserved. The Lagrangian does not depend on  $x$  (and so  $p_x$  is conserved), but it does depend on  $x$  and  $y$  (and so  $p_y$  is not conserved). If I make a transformation  $x \rightarrow x + \varepsilon$ , the derivative of  $x + \varepsilon$  does not change from the derivative of  $x$  ( $\dot{x}$ ). So, in  $(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + qby \dot{x})$ , if we transform  $x$ , it does not affect  $\dot{x}$ , and there is no  $x$  in the equation to affect. And changing  $x$ , obviously does not affect  $y$  or  $\dot{y}$ . On the other hand, if I make the transformation  $y \rightarrow y + \varepsilon$ , this will end up changing the term  $qby \dot{x}$ . What we have learned, is that in this gauge ( $p_x = m \dot{x} + qby$ ) is conserved. Let's take the case when  $p_x = 0$ . It will stay zero. If it starts out as some other quantity, it will stay that quantity as well.

$$m \dot{x} + qby = 0 \Rightarrow \dot{x} = -\frac{qb}{m}y, \text{ This tells us that the x-component of velocity is proportional to } y.$$

Now let us work out the theory with the other gauge ( $A_x = 0$  and  $A_y = -bx$ ) (Gauge 2).

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q[\vec{A}_i \cdot \vec{v}_i] = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q[A_y \dot{y}], \text{ (because } A_x = 0\text{).}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - qbx \dot{y}, \text{ Now let's calculate the canonical momenta conjugate to } x \text{ and } y.$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x}, \quad p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m \dot{y} - qbx, \text{ Which one of these is conserved? } p_y!$$

$$m \dot{y} - qbx = 0 \Rightarrow \dot{y} = \frac{qb}{m}x, \text{ This tells us that the x-component of velocity is proportional to } y.$$

So, without having to solve any equations, but rather by just writing down the conservation laws, we found out that the x-component of velocity is proportional to  $-y$ , and the y-component of velocity is proportional to  $x$  (circular motion).

Let's consider a particle moving in a circle around the origin. If we call the angle from the radius  $\theta$ , and the distance from the origin  $r$ , then  $x = r \cos \theta$  and  $y = r \sin \theta$ . Assuming an angular velocity:  $\omega$  (radians per second), then  $x = r \cos(\omega t)$  and  $y = r \sin(\omega t)$ . Let's check that this solves our equation.

$\dot{x} = -r\omega \sin(\omega t) = -\omega(y)$ ,  $\dot{y} = r\omega \cos(\omega t) = \omega(x)$ . So, we've shown that the x-component of velocity is proportional to  $-y$ , and the y-component of velocity is proportional to  $x$ . In order to find out what  $\omega$  is, we just have to compare the equations.

$$\dot{x} = -\frac{qb}{m}y = -\omega y \Rightarrow \omega = \frac{qb}{m},$$

$$\dot{y} = \frac{qb}{m}x = \omega x \Rightarrow \omega = \frac{qb}{m}.$$



Notice that we were able to write the canonical momentum as  $p_x = m \dot{x} + qby$  (Gauge 1).

And also as  $p_y = m \dot{y} - qbx$  (Gauge 2). Because the physics doesn't rely on which gauge you use, you can use gauge 1 to figure out what  $x$  does, and gauge 2 to figure out what  $y$  does. The circular motion we studied occurred when we set either one of these momenta equal to zero. Again, we are imagining this magnetic field to be completely uniform. So, every place in the magnetic field is equal force wise. Imagine another particle with a circular motion next to our original orbiting particle. Let us calculate if our above calculated momenta are still conserved. To that, we add a term to each location (the central coordinate of the circle  $(x_0, y_0)$ )...

$$x = r \cos(\omega t) + x_0$$

$$y = r \sin(\omega t) + y_0$$

Let's now calculate what the canonical momentum is along the x-axis (Gauge 1).

$$p_x = m \dot{x} + qby$$

$$= m[-r\omega \sin(\omega t)] + (\omega m)r \sin(\omega t) + qby_0, \text{ (substituting from } \omega = \frac{qb}{m} \text{ above)}$$

$$= -rm\omega \sin(\omega t) + \omega mr \sin(\omega t) + qby_0 = qby_0. \text{ And } p_x \text{ is conserved. And similarly, } p_y = -qbx_0.$$

So, the physical meaning of the momenta is entirely different than what is expected. The two gauges, which are the conserved canonical momentum, or simply proportional to the initial starting location of the circle around which the particles orbit. And what the conservation tells you is that the central location of the circle does not move.

Let's put back the electric field...

We're going to put it back in the X direction. It will not be the case anymore that  $p_x$  will be conserved.

Let's define the magnitude of the electric field to be notated as:  $|\vec{E}| = E$

That means some potential that looks like...

$$v = -Ex$$

$$-\frac{\partial v}{\partial x} = E_x = E, \quad -\frac{\partial v}{\partial y} = 0.$$

Lagrangian becomes...

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - qbx \dot{y} - (-qEx) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - qbx \dot{y} + qEx, \text{ } qEx \text{ is now the electric field force (energy).}$$

This does not affect the canonical momentum equations (because  $qEx$  does not depend upon velocity). However, there are no longer conserved. The time derivative of  $p_x$  is equal to the  $x$  derivative of  $qEx$ .

Why? Because of Lagrangian's equations...

$$p_x = m \dot{x} + qby$$

$$\frac{d}{dt}p_x = m \ddot{x} + qb \dot{y}$$

$$\frac{d}{dt}p_x = \frac{\partial \mathcal{L}}{\partial x} = qE = m \ddot{x} + qb \dot{y}$$

What about the equation for  $p_y$  in the other gauge (Gauge 2)? That is unchanged because I have not put any force in the  $y$  direction...

$$\frac{d}{dt}p_y = m \ddot{y} - qb \dot{x} = 0.$$

Now we want to look for a particular type of solution to the equations. For simplicity, let's look for a solution with no acceleration. That means the velocity has to be constant, but it can't be ANY constant. So, let us remove the terms which include acceleration from our time derivatives of the canonical momenta...

$$qE = qb \dot{y}, \text{ and } qb \dot{x} = 0.$$

This tells us that the movement in the  $x$  direction must be zero. However, from our other equation, we see that the movement in the  $y$  direction is equal to  $\frac{E}{b}$ , the ratio of the electric to the magnetic field.

Notice that the movement IS constant. We were correct to look for a solution that had no acceleration. And what we're left with is a very special velocity, namely a velocity in the  $y$  direction, with uniform motion equal to the ratio of the electric to the magnetic field. This is interesting because we put the electric force in pointing in the  $x$  direction. A bit of a surprise. That is called the Hall effect.

If you have a magnetic field and no electric field, is there any solution with no acceleration? Motion in a circle has acceleration. Nonetheless, there are solutions with no solution, namely a circle with zero radius where the particle just sits still. That is a solution of a Lorentz force law. No acceleration, no velocity.

However, when you turn on the electric field in the  $x$  direction, you get movement along the  $y$ -axis. That is called the Hall effect. So, if you place a bunch of charged particles on a plane with no kinetic energy, that also has a magnetic field imposed on the plane perpendicularly, the particles will not move.

However, if you then add an electric field parallel to the plane, the particles will then move along the plane in a direction perpendicular to the electric field, and with a velocity that is not proportional to the charge of the particle, but rather proportional to the amplitude of the electric field divided by the magnetic field.

## Poisson Brackets

And now for another form of mechanics which is an abstraction of the Hamiltonian form.

**Poisson brackets.** There would be very little point in teaching this if it were not so central to quantum mechanics.

The two Hamiltonian equations of motion are...  $\frac{\partial H}{\partial p_i} = \dot{q}_i$  and  $\frac{\partial H}{\partial q_i} = -\dot{p}_i$ . So, for each component  $i$ , we have two equations. We have twice as many equations as with the Hamiltonian formulation, but they are only first order differential equations. Let us take some function of velocity and position to consider in this context.  $A(p, q)$ , imagine this well-defined function moving through a phase space. as a particle moves through the phase flow, the value of the function changes with time. That is not to say that the function definition changes with time, but rather that the particle takes on differing values as it propagates through the phase space. So how does  $A$  change with time through a trajectory? We take the time derivative of the function along the trajectory, assuming that  $A$  has no explicit time dependency.

$\dot{A} = \frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i$ , (when we rate the equations in this way, the reader can assume it means to sum the equation over  $i$ )

Now let us substitute the Hamiltonian equations from above...

$$\dot{A} = \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i}$$

One could use Poisson Brackets on any two arbitrary functions. In the second we will be using  $A$  and  $H$ . But now, let's look at it more generally.

$\{A, B\} = \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$ , The elements of a Poisson bracket are not commutative. Instead...

$\{A, B\} = -\{B, A\}$ , So, Poisson brackets are anti-symmetric for any two functions.

So,  $\frac{d}{dt}A(p, q) = \{A, H\}$ . (let's only consider one  $p$  and one  $q$ , so we can stop writing the index  $i$ ).

Let's consider the extremely simple case of  $A = 1$ . In this instance, the function is unchanging throughout the phase space. Therefore, the time derivative is merely 0. Let's check it...

$$\{1, H\} = \frac{\partial 1}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial 1}{\partial p} \frac{\partial H}{\partial q} = 0 \cdot \frac{\partial H}{\partial p} - 0 \cdot \frac{\partial H}{\partial q} = 0.$$

Let's instead consider the case of  $A = p$ .

$$\begin{aligned} \frac{d}{dt}p &= \{p, H\} = \frac{\partial p}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial p}{\partial p} \frac{\partial H}{\partial q} \\ &= 0 \cdot \frac{\partial H}{\partial p} - 1 \cdot \frac{\partial H}{\partial q} \quad \left( \frac{\partial p}{\partial q} = 0, \text{ because } p \text{ and } q \text{ are independent variables perpendicular to each other in space} \right). \end{aligned}$$

$$= -\frac{\partial H}{\partial q} \text{ (and this is one of Hamilton's equations)}$$

Let's instead consider the case of  $A = q$ .

$$\begin{aligned} \frac{d}{dt} q &= \{q, H\} = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} \\ &= 1 \cdot \frac{\partial H}{\partial p} - 0 \cdot \frac{\partial H}{\partial q} \\ &= \frac{\partial H}{\partial p} \text{ (and this is the other of Hamilton's equations)} \end{aligned}$$

As you can see, we are reducing all of classical mechanics to an algebraic structure involving Poisson brackets.

Let's review a few properties that Poisson brackets have that are actually sufficient to recover any Poisson bracket you might be interested in.

$$\{A, B\} = -\{B, A\}$$

$$\{p, p\} = -\{p, p\} \Rightarrow \{p, p\} = \{q, q\} = 0. \text{ (any quantity that is equal to its negative must be equal to 0)}$$

$$\{p_i, p_j\} = \{q_i, q_j\} = 0.$$

$$\{q, p\} = \frac{\partial q}{\partial q} \frac{\partial p}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial p}{\partial q} = 1,$$

$$\{q_i, p_j\} = 0, \text{ if } i \neq j, \{q_i, p_j\} = \delta_{ij} = (0 \text{ if } i \neq j, 1 \text{ if } i = j) \text{ (Kronecker Delta)}$$

$$\{p, q\} = \frac{\partial p}{\partial q} \frac{\partial q}{\partial p} - \frac{\partial p}{\partial p} \frac{\partial q}{\partial q} = -1,$$

$$\{p_i, q_j\} = 0, \text{ if } i \neq j, \{p_i, q_j\} = -\delta_{ij} = (0 \text{ if } i \neq j, -1 \text{ if } i = j)$$

The symplectic structure of quantum mechanics says that whenever you switch  $q$  and  $p$ , you end up with a negative sign.

$\{p, F(q, p)\} = \frac{\partial p}{\partial q} \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q} \frac{\partial p}{\partial p} = 0 - \frac{\partial F}{\partial q} = -\frac{\partial F}{\partial q}$ . Taking the Poisson brackets of an arbitrary function with respect to  $p$  is the same as taking the derivative of that function with respect to  $q$  with a minus sign.

With many variables...  $\{p_i, F(q, p)\} = -\frac{\partial F}{\partial q_i}$ .

$\{q_i, F(q, p)\} = +\frac{\partial F}{\partial p_i}$ . So taking Poisson brackets with  $q$ s and  $p$ s is basically the act of taking a derivative.

One big take away: the time derivative of anything, is equal to the Poisson bracket of that thing with  $H$  ( $A = \{A, H\}$ ).

Linearity of the Poisson brackets with respect to one of the entries:

$$\{\alpha A, B\} = \alpha \{A, B\} \text{ where } \alpha \in \mathbb{R}$$

$$\{A + C, B\} = \{A, B\} + \{C, B\} \text{ where } \alpha \in \mathbb{R}.$$

These properties are almost enough to determine the Poisson brackets of any function of  $p$  and  $q$ , with any other function of  $p$  and  $q$ .

The last element you need is a product rule...

$$\begin{aligned} \{AB, C\} &= \frac{\partial AB}{\partial q} \frac{\partial C}{\partial p} - \frac{\partial AB}{\partial p} \frac{\partial C}{\partial q} \\ &= A \frac{\partial B}{\partial q} \frac{\partial C}{\partial p} + B \frac{\partial A}{\partial q} \frac{\partial C}{\partial p} - A \frac{\partial B}{\partial p} \frac{\partial C}{\partial q} - B \frac{\partial A}{\partial p} \frac{\partial C}{\partial q} \\ &= A \left( \frac{\partial B}{\partial q} \frac{\partial C}{\partial p} - \frac{\partial B}{\partial p} \frac{\partial C}{\partial q} \right) + B \left( \frac{\partial A}{\partial q} \frac{\partial C}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial C}{\partial q} \right) = A \{B, C\} + B \{A, C\}. \end{aligned}$$

## Lecture 9

### Poisson Brackets

Phase space has a structure to it, it's not just a bunch of  $qs$  and  $ps$ , but rather it's a bunch of  $qs$  and  $ps$  with a certain kind of structure to it. Spaces have structures, the structures are the things which are invariant under the various transformations that you might want to do. An example of a structure that might be on a space, if you know anything about curved spaces and differential geometry, the metric structure of metrical spaces (of Riemann spaces) are part of the structure of those spaces. They are characterized by a metric in that space, a distance formula between neighboring points. Poisson structures are different from metric structures in that they are more abstract, and they were created largely to describe the structure of the space, and its properties under transformations which change the coordinates. The coordinates of the phase space are not just the  $qs$ , but include the  $ps$ . So, a basic question you might have is how might one conduct transformations in this phase space that maintains the  $qs$  and  $ps$ , while maintaining the structure of the phase space.

Poisson brackets are useful for describing flows on phase space. One type of flow in phase space is the motion of particles through the space with time. All the particles propagate with time under the influence of a particular Hamiltonian. However, there are other flows that can be examined in phase space for other purposes; for example symmetries. The problem with discussing symmetries as it relates to classical mechanics is the lack of examples which give a large range of situations. In classical mechanics, we're generally working with rotational and translational symmetries.

In advance of our examining symmetries in phase space, let's look at it in Cartesian space. If we think about rotation in the space, we can think of it in two ways. We can either imagine an object rotating, or equivalently locating the origin of this space in the center of the object, and rotating the coordinate system. Either way, you can divide up the rotation into infinitely small fractions of the rotation, and see the large rotation as merely a summation of all of these smaller fractions. You can imagine the moving of the coordinate system as a type of flow. One can similarly apply this idea to translation. In phase space, similar flows occur, but instead of just adjusting directional coordinates ( $qs$ ), we are also manipulating  $qs$  and  $ps$ . And these flows in phase space are described by Poisson brackets.

A more complicated version of linearity:  $\{\alpha A + \beta B, C\} = \alpha\{A, C\} + \beta\{B, C\}$

While Poisson brackets in classical mechanics are commutative in how they are multiplied by functions, they are not necessarily in quantum mechanics. So, an alternate version of the product rule which is more consistent is...

$$\{AB, C\} = A\{B, C\} + \{A, C\}B$$

A linguistic note: When physicists use the term scalar, they need a property which does not change under the rotation of space. This differs from its use in mathematics.

Because the definition of Poisson brackets includes both  $\frac{\partial}{\partial p}s$  and  $\frac{\partial}{\partial q}s$ , if you place to functions within the brackets, both of which only depend upon a single variable, you get...  $\{F(q), G(q)\} = 0$  and  $\{F(p), G(p)\} = 0$ .

So, we have an algebraic system of Poisson brackets which characterize the relationship between  $qs$  and  $ps$  in phase space. Let's add to that, another postulate about the time evolution of a system that can be completely derived from Hamilton's equations. In fact, we've directed a couple of times previously in this course...

$\dot{A}(p, q) = \{A, H\}$ , This means that the time derivative of the function in phase space, or the speed with which the  $qs$  and  $ps$  change in phase space is equal to the Poisson brackets of that function with the Hamiltonian.  $\dot{A}$  is the derivative with respect to time to the function on the phase space, and is thought

of as applying the flow to the function for a particular particle.

Let's see what happens if our Hamiltonian happens to be  $\frac{p^2}{2m}$ . Then, we apply this Hamiltonian to  $\dot{p}$  within the Poisson brackets.  $\dot{p} = \{p, H\} = \left\{p, \frac{p^2}{2m}\right\} = 0$  (because, as stated earlier, the possible bracket of  $p$ , with any other function which is solely a function of  $p$  equals zero).  $\dot{q} = \{q, H\} = \left\{q, \frac{p^2}{2m}\right\} = \frac{\partial}{\partial p} \left(\frac{p^2}{2m}\right) = \frac{p}{m}$ . So, without knowing Hamilton's equations, we are able to derive these properties using Poisson brackets.

## Symmetries

Symmetries are transformations of a system which do not change the dynamics.

Up until now, the symmetries we have studied have been transformations in terms of one of the variables  $q$ , but not of  $p$ . Such as rotations in space...

$$Q = Q(q).$$

So, are there symmetries in nature which include both  $qs$  and  $ps$ . And if such symmetries do exist, what are the rules/laws which allow you to manipulate the  $qs$  and  $ps$ , while simultaneously preserving the laws of classical mechanics. We want rules (transformations on phase space) which maintain a form we can recognize, namely the Poisson algebra.

For example, let's try making an arbitrary transformation and see how it affects things.

Let's define the following transformations...  $P = 2p$  and  $Q = 2q$ . So, the question is whether or not these transformations preserve the Poisson structures.

$$\{P, Q\} = \{2p, 2q\} = 4, \text{ This does not maintain our Poisson structure.}$$

Instead, how about  $P = \frac{p}{2}$  and  $Q = 2q$ .

$$\{P, Q\} = \left\{\frac{p}{2}, 2q\right\} = 1, \text{ This does maintain our Poisson structure.}$$

It's not necessarily a symmetry in a physical system, but it is another set of coordinates which is exactly the same Poisson bracket relations as the original. In the phase space, we would see the initial transformation (which did not maintain Poisson structure) as an attempt to stretch the phase space both vertically and horizontally. The second transformation (which didn't maintain the Poisson structure) was an attempt to squeeze the space horizontally while simultaneously stretching vertically. When only working with one  $q$  and  $p$ , this has a conservation of area.

$$\text{Instead, how about } P = \cos\theta \cdot p + \sin\theta \cdot q \text{ and } Q = -\sin\theta \cdot p + \cos\theta \cdot q$$

This will have the effect of rotating the phase space ( $P$  rotated into  $Q$ ), and should therefore maintain the area. So, we might suspect that it will preserve our Poisson structure.

$$\begin{aligned} \{Q, P\} &= \{q \cos\theta - p \sin\theta, p \cos\theta + q \sin\theta\} \\ &= \{q \cos\theta, p \cos\theta + q \sin\theta\} - \{p \sin\theta, p \cos\theta + q \sin\theta\} \\ &= -\{p \cos\theta + q \sin\theta, q \cos\theta\} + \{p \cos\theta + q \sin\theta, p \sin\theta\} \\ &= -\{p \cos\theta, q \cos\theta\} - \{q \sin\theta, q \cos\theta\} + \{p \cos\theta, p \sin\theta\} + \{q \sin\theta, p \sin\theta\} \\ &= \{q \sin\theta, p \sin\theta\} - \{p \cos\theta, q \cos\theta\} \\ &= \left(\frac{\partial(q \sin\theta)}{\partial q} \frac{\partial(p \sin\theta)}{\partial p} - \frac{\partial(p \sin\theta)}{\partial q} \frac{\partial(q \sin\theta)}{\partial p}\right) - \left(\frac{\partial(p \cos\theta)}{\partial q} \frac{\partial(q \cos\theta)}{\partial p} - \frac{\partial(q \cos\theta)}{\partial q} \frac{\partial(p \cos\theta)}{\partial p}\right) \\ &= \left(\frac{\partial(q \sin\theta)}{\partial q} \frac{\partial(p \sin\theta)}{\partial p} - 0 \cdot \frac{\partial(q \sin\theta)}{\partial p}\right) - \left(0 \cdot \frac{\partial(q \cos\theta)}{\partial p} - \frac{\partial(q \cos\theta)}{\partial q} \frac{\partial(p \cos\theta)}{\partial p}\right) \\ &= \frac{\partial(q \sin\theta)}{\partial q} \frac{\partial(p \sin\theta)}{\partial p} + \frac{\partial(q \cos\theta)}{\partial q} \frac{\partial(p \cos\theta)}{\partial p} = \sin\theta \cdot \sin\theta + \cos\theta \cdot \cos\theta = 1. \text{ This does maintain our Poisson} \end{aligned}$$

structure.

The family of transformations which preserve the Poisson structure are called **Canonical Transformations**. Notice that these are not gauge transformations. These are not redundancies of a

description, they are changes of a description to new coordinates. When you are talking about canonical transformations, you're talking about a change in description of the system, but not a change in the physics of that system.

Let's talk about how we build up the infinitesimal transformations of a symmetry, until it adds up to be the whole transformation. What do we mean by an infinitesimal transformation? It means that the new system  $Q$ , is only different from the old system  $q$  by a little amount:  $\delta q(p, q)$ .

$$Q = q + \delta q(p, q)$$

This just means that deviation from a trivial transformation is infinitesimal ( $\delta q(p, q)$ ). In practice, what it means is that in their formulas, we will drop things proportional to quadratic and higher powers of the deltas (presumably because if they are less than 1, higher powers while only decrease the value?). It means that small quantities, are quantities that are so small that we can drop higher powers of them. that is what we will mean by infinitesimal, that we can drop higher powers of them. We can always generate larger transformations by adding together smaller transformations for which we have dropped higher powers of their infinitesimals, but then we have to combine them together which can become complicated. We can test out very easily one of the rules for infinitesimal canonical transformations.

$P = p + \delta p$ , (for efficiency, this text will assume that each  $\delta p = \delta p(p, q)$ , we will also suppress ( $p = p_i$ ) the indices until the last step).

Let us write down the condition that this be a canonical transformation.

The interesting one here is...

$$\begin{aligned} \{Q, P\} &= \{q + \delta q, p + \delta p\} = \{q, p + \delta p\} + \{\delta q, p + \delta p\} = \{q, p\} + \{q, \delta p\} + \{\delta q, p\} + \{\delta q, \delta p\} \\ &= \{q, p\} + \{\delta q, p\} + \{q, \delta p\} + 0. \text{ Now, we want } \{Q, P\} = \{q, p\}, \text{ which means we need} \\ \{\delta q, p\} &= -\{q, \delta p\}. \text{ This is necessary and sufficient. Let's see the construction...} \end{aligned}$$

Let's call  $\delta q = \varepsilon \{q, G(p, q)\}$  where  $G$  is the generator of the canonical momentum.

And  $\delta p = \varepsilon \{p, G(p, q)\}$ . Every infinitesimal canonical transformation is characterized by a function  $G(p, q)$ , which itself defines it a flow.

Let's now prove the theory that if  $\delta q$  and  $\delta p$  are obtained by Poisson brackets with a generator  $G$ , then it will always be true that  $\{\delta q, p\} = -\{q, \delta p\}$  (it can also be proved in the opposite direction which means both conditions are necessary and sufficient to show the other).

First of all,  $\delta q = \varepsilon \{q, G(p, q)\} = \varepsilon \frac{\partial G}{\partial p}$  and  $\delta p = \varepsilon \{p, G(p, q)\} = -\varepsilon \frac{\partial G}{\partial q}$ , with substitution...

$$\left\{ \varepsilon \frac{\partial G}{\partial p}, p \right\} = -\left\{ q, -\varepsilon \frac{\partial G}{\partial q} \right\} \Rightarrow \varepsilon \left\{ \frac{\partial G}{\partial p}, p \right\} = \varepsilon \left\{ -\frac{\partial G}{\partial q}, q \right\} \Rightarrow \left\{ \frac{\partial G}{\partial p}, p \right\} = \left\{ -\frac{\partial G}{\partial q}, q \right\} \Rightarrow -\frac{\partial}{\partial q} \frac{\partial G}{\partial p} = -\frac{\partial}{\partial p} \frac{\partial G}{\partial q}. \text{ And since it does not matter the order with which you partially differentiate something, these two terms are indeed equal. Q.E.D.}$$

What we have found is that flows that are generated by Poisson bracketing with respect to a generator always defines canonical transformations. A special case of this is the Hamiltonian flow. Let's look again at these equations...

$$\delta q = \varepsilon \{q, G(p, q)\} = \varepsilon \frac{\partial G}{\partial p} \text{ and } \delta p = \varepsilon \{p, G(p, q)\} = -\varepsilon \frac{\partial G}{\partial q}$$

Supposing we were talking about the time motion of the system. The time motion of a system is

$$\dot{q} = \{q, H\} \text{ but if I instead write } \dot{q} \text{ as } \frac{\delta q}{\delta t}, \text{ and I just think of } \delta t \text{ as } \varepsilon,$$

$\frac{\delta q}{\delta t} = \{q, H\} \Rightarrow \delta q = \delta t \{q, H\} = \varepsilon \{q, H\}$ , then the Hamiltonian flow takes on the form we just worked with. Hamiltonian evolution of a system is a canonical transformation. In other words, the coordinate transformation of  $q$ s and  $p$ s which are generated by the actual flow of the system, is itself a special case of a canonical transformation. Furthermore, all canonical transformations can be generated by picking some generator (as it were the Hamiltonian), imagining in our mind a Hamiltonian which caused the flow which took the points from one place in the phase space to another, that defines the most general

class of canonical transformations.

Above, we failed to reintroduce the indices. This can be a homework exercise.

If the transformations we are allowed to do, the ones that preserve the structure of mechanics are canonical, with a given system and a given Hamiltonian, what is the subclass of those that are called symmetries? Symmetries are the canonical transformations that do not change the Hamiltonian. This is similar to what we said about the Lagrangian. We said that the coordinate transformations that did not change the Lagrangian were symmetries. Now, the more general idea becomes: coordinate transformations in phase space that are canonical, and which do not change the energy are symmetries.

We can also say this in a geometric way. Imagine a two-dimensional phase space in which the Hamiltonian corresponds to a flow. And it is also a canonical transformation. The infinitesimal smalltime flow of any particles in the phase space are generated by the Hamiltonian, and the full transformation over a period of time is canonical. Now, let us imagine another quantity, a generator  $G$ . It generates a different flow, but not the time differentiated flow. It moves points through the phase space in some other way. This can be imagined as flow lines traveling in a direction nonparallel to the Hamiltonian flow. This would often have the effect of changing the energy of the system. However, imagine that the Hamiltonian, and  $G$  flows relate in just such a way that the amount of energy in the system is not affected. In this case, it is called the symmetry. What is the condition for that? Here is how  $q$  and  $p$  change under the flow;  $\delta q = \varepsilon \{q, G(p, q)\} = \varepsilon \frac{\partial G}{\partial p}$  and  $\delta p = \varepsilon \{p, G(p, q)\} = -\varepsilon \frac{\partial G}{\partial q}$ .

Let's take a general function for  $q$  and  $p$ , and see how it changes along this flow.

$\delta A = \frac{\partial A}{\partial q} \delta q + \frac{\partial A}{\partial p} \delta p$ , and with substitution from above we can rewrite this as...

$\delta A = \frac{\partial A}{\partial q} \left( \varepsilon \frac{\partial G}{\partial p} \right) + \frac{\partial A}{\partial p} \left( -\varepsilon \frac{\partial G}{\partial q} \right) = \varepsilon \left( \frac{\partial A}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial G}{\partial q} \right) = \varepsilon \{A, G\}$ , This is the exact formula used for the time derivative of things when  $G$  was the Hamiltonian. The change of functions along the Hamiltonian flow is just the Poisson bracket with a Hamiltonian. The change of any arbitrary function along any flow is just the Poisson bracket with the generator of the flow. That's what these generators do. The Poisson bracket with the generator create the flow and they give you the derivatives of the way the functions change along the flow. What is it mean that the energy does not change in the direction of the flow  $G$ ? It says that  $\{H, G\} = 0$ . It means that the Hamiltonian does not change in the direction of  $G$ . We can verify this by substituting  $H$ , in our previous equation.

$$\delta H = \frac{\partial H}{\partial q} \left( \varepsilon \frac{\partial G}{\partial p} \right) + \frac{\partial H}{\partial p} \left( -\varepsilon \frac{\partial G}{\partial q} \right) = \varepsilon \left( \frac{\partial H}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial G}{\partial q} \right) = \varepsilon \{H, G\}.$$

But this must also mean that  $\{G, H\} = 0$ , and that the value of  $G$  does not change in the direction of the Hamiltonian. And since,  $\{G, H\} = -\{H, G\}$ , it tells us that  $G$  does not change with time ( $G$  is conserved). This creates a symmetrical relationship between the two generators. This relationship is much easier to conceive of in quantum mechanics because there are so many more examples, concrete examples to examine.

One real life example is angular momentum:  $H = \frac{p^2}{2m} + F(x^2 + y^2)$ ,  $G = xp_y - yp_x$

Homework exercise, show that:  $\{G, H\} = 0$ ,  $\left\{ xp_y - yp_x, \frac{p^2}{2m} + F(x^2 + y^2) \right\} = 0$

We're going to show it with a free particle, as it is a bit difficult to show it in general. We are also going to set the mass equal to 1.  $G$  is the angular momentum, and  $H$  as usual is the Hamiltonian (the energy).

$$\begin{aligned} H &= \frac{p_x^2}{2} + \frac{p_y^2}{2}, & G &= xp_y - yp_x \\ \{G, H\} &= \left\{ \frac{p_x^2}{2} + \frac{p_y^2}{2}, xp_y - yp_x \right\} = \left\{ \frac{p_x^2}{2}, xp_y - yp_x \right\} + \left\{ \frac{p_y^2}{2}, xp_y - yp_x \right\} \\ &= \left\{ \frac{p_x^2}{2}, xp_y \right\} + \left\{ \frac{p_x^2}{2}, -yp_x \right\} + \left\{ \frac{p_y^2}{2}, xp_y \right\} + \left\{ \frac{p_y^2}{2}, -yp_x \right\} \end{aligned}$$

$= \left\{ \frac{p_x^2}{2}, xp_y \right\} + 0 + 0 + \left\{ \frac{p_y^2}{2}, -yp_x \right\}$ , (Poisson brackets in which you pair a momenta of one variable with a different variable is equal to zero, the same occurs when pairing a momenta with any other momenta)

$$\begin{aligned}
 &= \left\{ yp_x, \frac{p_y^2}{2} \right\} - \left\{ xp_y, \frac{p_x^2}{2} \right\} \\
 &= y \left\{ p_x, \frac{p_y^2}{2} \right\} + p_x \left\{ y, \frac{p_y^2}{2} \right\} - x \left\{ p_y, \frac{p_x^2}{2} \right\} - p_y \left\{ x, \frac{p_x^2}{2} \right\} \\
 &= y \cdot 0 + p_x \left\{ y, \frac{p_y^2}{2} \right\} - x \cdot 0 - p_y \left\{ x, \frac{p_x^2}{2} \right\} = p_x \frac{\partial}{\partial p} \left( \frac{p_y^2}{2} \right) - p_y \frac{\partial}{\partial p} \left( \frac{p_x^2}{2} \right) = p_x p_y - p_y p_x = 0.
 \end{aligned}$$

So, this is in example of two generators that have vanishing Poisson brackets between them. The implication is that is the Hamiltonian is  $\frac{p_x^2}{2} + \frac{p_y^2}{2}$ , then the angular momentum  $xp_y - yp_x$  is conserved. Interestingly, if we were to invent a new phase space in which the Hamiltonian were  $xp_y - yp_x$ , we would find that there would be another conserved generator whose value was  $\frac{p_x^2}{2} + \frac{p_y^2}{2}$ . And, again, the idea of the symmetry is a generator which does not change the energy of the system (the Hamiltonian).

So let us work out the example in which  $H = xp_y - yp_x$ .

$$\begin{aligned}
 \dot{x} &= \{x, H\} = \{x, xp_y\} - \{x, yp_x\} = \{yp_x, x\} - \{xp_y, x\} = y\{p_x, x\} + p_x\{y, x\} - x\{p_y, x\} - p_y\{x, x\} = y \cdot -1 + p_x \cdot 0 - \\
 \dot{y} &= \{y, H\} = x.
 \end{aligned}$$

These velocities would correspond with the particle traveling in a circle. As a homework assignment, show that  $\frac{p_x^2}{2} + \frac{p_y^2}{2}$  is conserved.

From the audience: How do you find the  $p$ s when given the velocities of a system?

Answer: You start with Hamilton's equation, and you work it out using Poisson brackets.

$$H = \frac{p^2}{2m} + U(x)$$

$$\dot{x} = \{x, H\} = \frac{p}{m}$$

$$\dot{p} = \{p, H\} = \frac{\partial U}{\partial x}$$