MATH 1271: Calculus I

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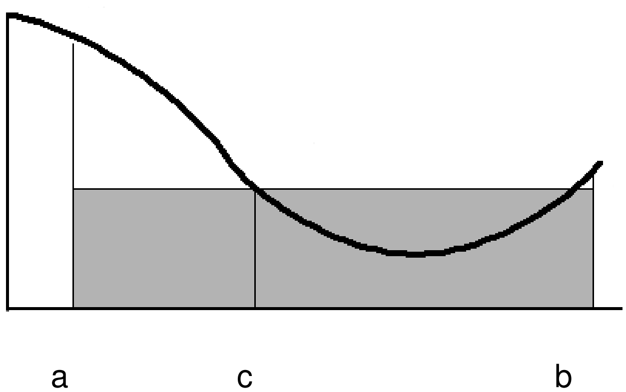
6.5 - Average Value of a Function

Review:

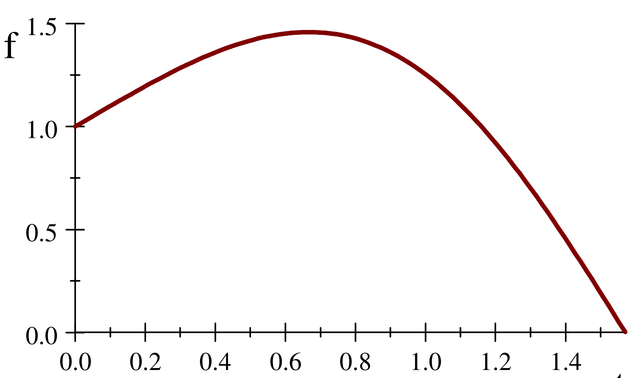
Recall how to find the average of some numbers. Given $\{5,\ 1,\ 2,\ 9,\ 27,\ \pi\}$, we average them by summing them together, and dividing by the number of values. $Avg = \frac{1}{6}(5+1+2+9+27+\pi) \approx 7.86.$

Let's define the analogous average for a continuous function using integrals.
Average of Value of $f(x)$ is defined as: $f_{avg} = \frac{1}{\text{"number" of values}}(\text{"sum" of values}) = \frac{1}{b-a}\int_a^b f(x)dx.$

Mean Value for Integrals: If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that: $f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx.$ Put another way: $\int_a^b f(x)dx = f(c)(b-a).$



Example 1. Find the average value of $f(t) = e^{\sin t}\cos t$ on the interval $[0, \frac{\pi}{2}]$.



First integrate the function over the interval:

$\int_0^{\frac{\pi}{2}} (e^{\sin t} \cos t) dt$

$u = \sin t, \qquad du = \cos t\, dt.$

So: $\int_0^{\frac{\pi}{2}} (e^{\sin t} \cos t) dt = \int_{t=0}^{t=\frac{\pi}{2}} e^u du$

For bounds of integration $t = 0$ when $u = \sin 0 = 0$, and $t = \frac{\pi}{2}$ when $u = \sin \frac{\pi}{2} = 1$.

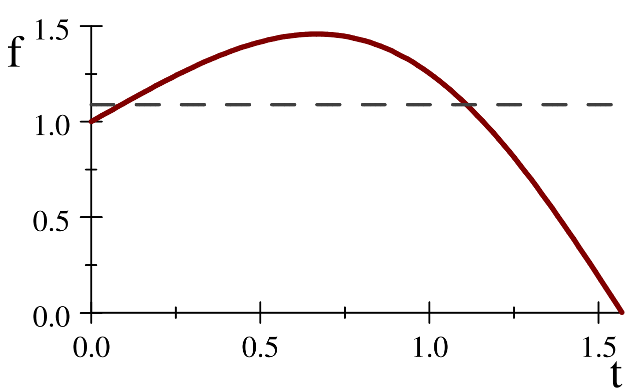
So, $\int_{t=0}^{t=\frac{\pi}{2}} e^u du = \int_{u=0}^{u=1} e^u du$

$[e^u]|_0^1 = e - 1.$

Are we done?

"Find the average value of $f(t) = e^{\sin t} \cos t$ on the interval $[0, \frac{\pi}{2}]$."

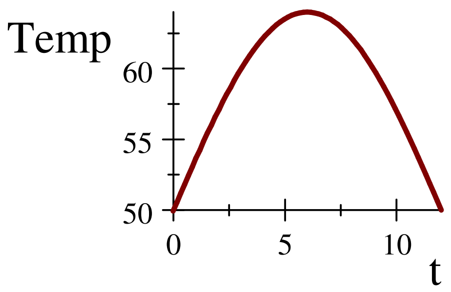
Average value of the function: $\frac{1}{\frac{\pi}{2} - 0}(e-1) = \frac{2}{\pi}(e-1) \approx 1.094.$



$e^{\sin t} \cos t$ and 1.094

Example 2. In Minneapolis, the temperature t hours after 9 AM (in $°F$) is modeled by the function: $T(t) = 50 + 14 \sin(\frac{\pi}{12} t).$

Find the average temperature during the period from 9 AM to 9 PM.



$50+14\sin(\frac{\pi}{12}t)$

What are our bounds of integration?

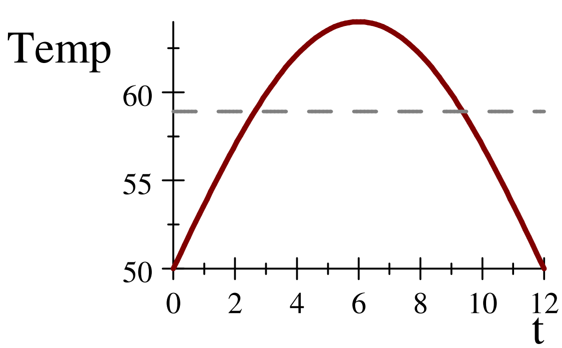
Average Temperature $= \frac{1}{12}\int_0^{12}(50+14\sin(\frac{\pi}{12}t))dt$

$= \frac{1}{12}[50t - 14(\frac{12}{\pi}\cos(\frac{\pi}{12}t))]_0^{12}$ (reverse trig derivative and reverse chain rule!)

$= \frac{1}{12}[600 - 14(\frac{12}{\pi}\cos\frac{\pi 12}{12}) - (0 - 14(\frac{12}{\pi}\cos\frac{\pi \cdot 0}{12}))]$

$= \frac{1}{12}(600 + 14(\frac{12}{\pi}) + 14(\frac{12}{\pi}))$

$= 50 + 28(\frac{1}{\pi}) = 50 + \frac{28}{\pi} \approx 58.9^{\circ}F.$



$50+14\sin\frac{\pi t}{12}$

Problem 8. Find the average value of $h(u) = (3-2u)^{-1}$ on the interval $[-1,1]$.

$$h_{avg} = \frac{1}{b-a}\int_a^b h(u)du = \frac{1}{1-(-1)}\int_{-1}^{1}(3-2u)^{-1}du = \frac{1}{2}\int_{-1}^{1}\frac{1}{3-2u}du$$

$$y = 3-2u, \qquad dy = -2du$$

Bounds of integration: When $u = -1$, we have $y = 3+2 = 5$. And when $u = 1$, we have $y = 3-2 = 1$.

$$h_{avg} = \frac{1}{2}\int_5^1 \frac{1}{y}\left(-\frac{1}{2}\right)dy = -\frac{1}{4}\int_5^1 \frac{1}{y}dy$$

$$= -\frac{1}{4}[\ln|y|]_5^1 = -\frac{1}{4}(\ln 1 - \ln 5) = \frac{1}{4}\ln 5.$$

Problem 13. If f is continuous and $\int_1^3 f(x)dx = 8$, **show** ("**show**" **means** ***prove***) that f takes on the value 4 at least once on the interval $[1,3]$.

By the Medium Value Theorem, since f is continuous on $[1,3]$, then there exists a number c in $[1,3]$ such that: $f(c) = f_{avg} = \frac{1}{3-1}\int_1^3 f(x)dx$.

However, we are given that $\int_1^3 f(x)dx = 8$, so $f(c) = 4$, so f takes on the value 4 at least once on the interval $[1,3]$.

Problem 14. Find the numbers b such that the average value of $f(x) = 2+6x-3x^2$ on the interval $[0,b]$ is equal to 3.

The requirement is that: $\frac{1}{b-0}\int_0^b f(x)dx = 3$.

Observe that the LHS of this equation is equal to:

$$\frac{1}{b}\int_0^b(2+6x-3x^2)dx = \frac{1}{b}[2x+3x^2-x^3]_0^b = 2+3b-b^2.$$

So we solve the equation: $2+3b-b^2 = 3 \quad \Rightarrow \quad b^2-3b+1 = 0$

$$\Rightarrow \quad b = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{3 \pm \sqrt{5}}{2} \approx \{0.382,\ 2.618\}.$$

Both values of b are solutions since they are positive (create an interval $[0, b]$)