MATH 1271: Calculus I

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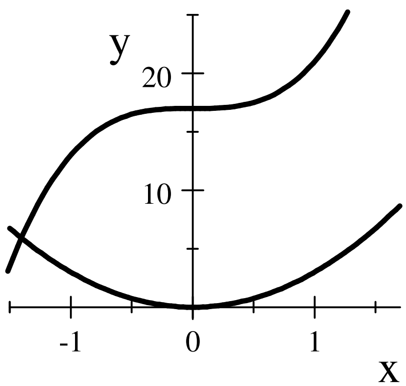
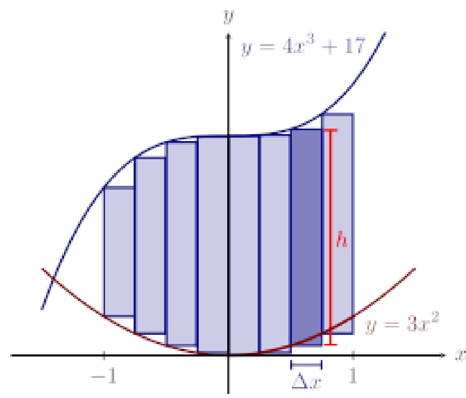
6.1 - Area Between Curves

Review:

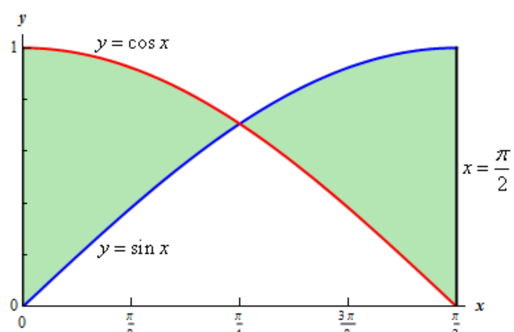




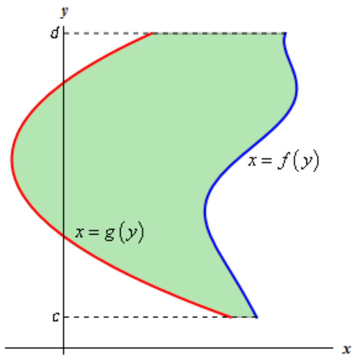
Approximating the Area Between Two Curves: $A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$

$A = \int_{x=a}^{x=b} [f(x) - g(x)]dx$

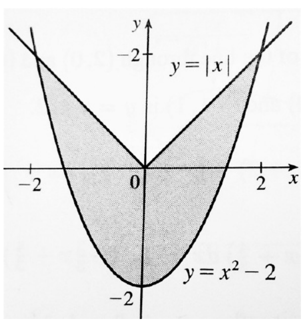
Finding the area between the curves when $f \geq g$ **for part of an interval**, **and** $f \leq g$ **for another part** (see image below): $A = \int_{x=a}^{x=b} |f(x) - g(x)|dx$.



Finding the area when the 2 curves f and g **are horizontally** (**instead of vertically**) **separated** (see image below). In order to be functions in y, we must get only one x value for each y (horizontal line test) or "$x = f(y)$" for our right curve and similarly for the left curve, $x = g(y)$, and the area between them is then: $A = \int_{y=c}^{y=d} [f(y) - g(y)]dy$.



Problem 26. Sketch the region enclosed by the curves: $y = |x|$ **and** $y = x^2 - 2$. **Then, find its area**.



What are our bounds of integration?

For $x > 0,\ |x| = x$, so the curves intersect when:

$$x = x^2 - 2 \quad \Rightarrow \quad 0 = x^2 - x - 2 \quad \Rightarrow \quad 0 = (x-2)(x+1) \quad \Rightarrow \quad x = 2.$$

And similarly for $x < 0,\ |x| = -x$, and we find $-x = x^2 - 2$ when $x = -2$.

Now before we start calculating the integral, notice we have symmetry (both $|x|$ and $x^2 - 2$ are even), so it is sufficient to find the area between the curves when x is greater than zero, and then double it.

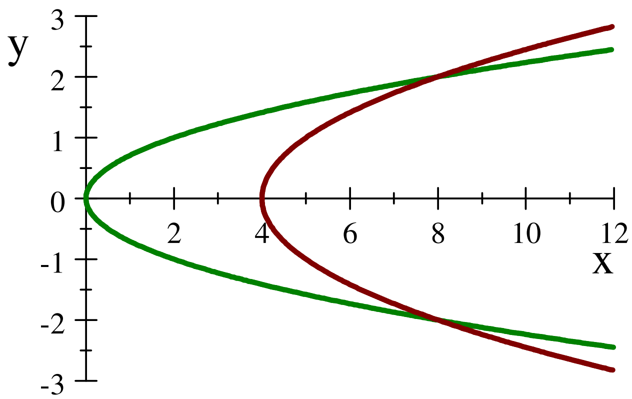
So, $A = 2\int_0^2 [x - (x^2 - 2)]dx = 2\int_0^2 (x - x^2 + 2)dx$

$$= 2[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x]_0^2 = 2\Big[\big(\tfrac{1}{2}(2)^2 - \tfrac{1}{3}(2)^3 + 2(2)\big) - \big(\tfrac{1}{2}(0)^2 - \tfrac{1}{3}(0)^3 + 2\cdot 0\big)\Big]$$

$$= 2(2 - \tfrac{8}{3} + 4) = \tfrac{20}{3}.$$

Problem 17 **Sketch the region enclosed by the curves**: $x = 2y^2$ **and** $x = 4 + y^2$. **Find its area**.

Note that both of these are parabolas. Of course, the role of x and y have been switched, so they are parabolas expanding to the right in the Cartesian coordinate plane. Also note that $4 + y^2$ is "lifted" (to the right) by 4, and $2y^2$ grows more quickly as y increases. So the image we might have in our mind is:



If we are going to know our bounds of integration, we will need to know where these intersect. Setting the equations equal to each other: $2y^2 = 4 + y^2 \quad \Rightarrow \quad y^2 = 4$ or $y = \pm 2$.

$$\int_{y=-2}^{2} \left(\text{Right} - \text{Left}\right) dy$$

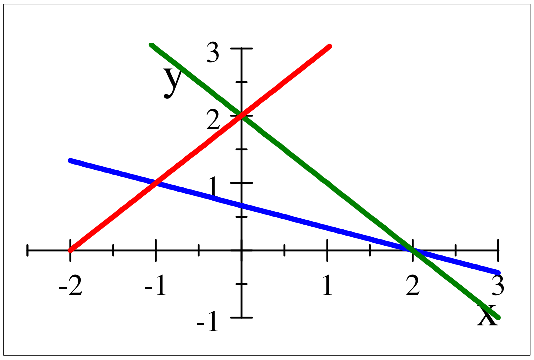
$$= \int_{y=-2}^{2} (4 + y^2 - 2y^2) dy = \int_{y=-2}^{2} 4 - y^2 dy$$

$$= [4y - \tfrac{1}{3} y^3]_{y=-2}^{2}$$

$$= (4 \cdot 2 - \tfrac{1}{3} 2^3) - \left(4(-2) - \tfrac{1}{3}(-2)^3\right) = \tfrac{32}{3}.$$

Problem 30. Use calculus to find the area of the triangle with the given vertices.

$(-1, 1)$, $(0, 2)$, $(2, 0)$,



Cut up the triangle into the positive part, and the negative part (since when integrating from left to right, this is where the upper functions change: from the increasing line, to the decreasing line).

Discover the functions which define the lines, so we can integrate the areas between them.

The slope of the upper-left line through $(-1,1)$ and $(0,2)$ is: $slope = m_g = \frac{rise}{run} = \frac{1}{1} = 1$,

....upper-right line through $(0,2)$ and $(2,0)$ is: $m_r = \frac{-2}{2} = -1$

...lower line through $(-1,1)$ and $(2,0)$ is: $m_b = \frac{-1}{3} = -\frac{1}{3}$,

Using point-slope form, the equation of the upper-left line through $(0,2)$ is: $y - 2 = x$.

...upper-right line through $(2,0)$ is: $y = -1(x-2)$;

...lower line through $(2,0)$ is: $y = -\frac{1}{3}(x-2)$;

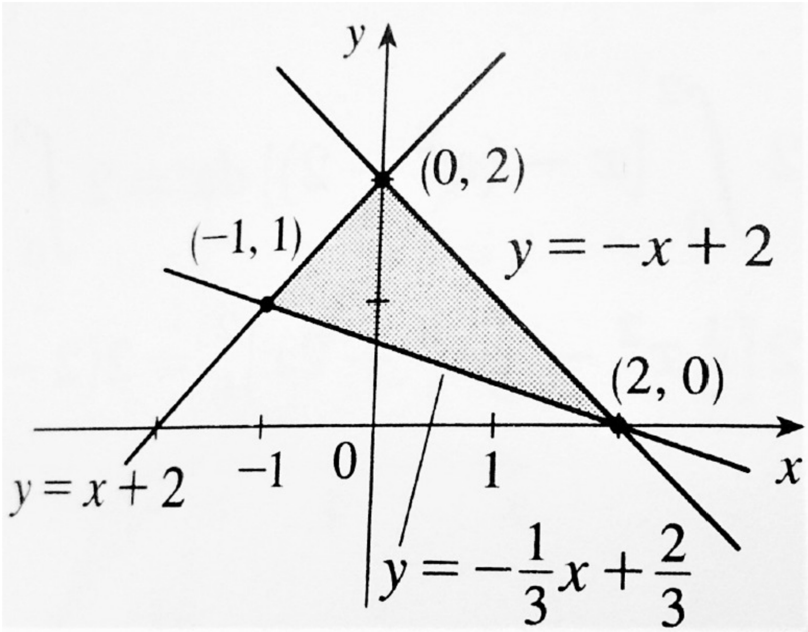
Then, putting together our integral we have:

$$A = \int_{-1}^{0}[(x+2) - (-\tfrac{1}{3}x + \tfrac{2}{3})]dx + \int_{0}^{2}[(-x+2) - (-\tfrac{1}{3}x + \tfrac{2}{3})]dx$$

$$= \int_{-1}^{0}(\tfrac{4}{3}x + \tfrac{4}{3})dx \ + \ \int_{0}^{2}(-\tfrac{2}{3}x + \tfrac{4}{3})dx$$

$$= \ [\tfrac{2}{3}x^2 + \tfrac{4}{3}x]_{-1}^{0} + [-\tfrac{1}{3}x^2 + \tfrac{4}{3}x]_{0}^{2}$$

$$= 0 - (\tfrac{2}{3} - \tfrac{4}{3}) + (-\tfrac{4}{3} + \tfrac{8}{3}) - 0 = 2.$$



Problem 32. Evaluate $\int_{-1}^{1} |3^x - 2^x| dx$ and interpret it as the area of a region. Sketch the region.

To rid ourselves of the absolute value sign, we must determine when $3^x - 2^x < 0$.

$$3^x < 2^x$$

$$\Rightarrow \quad x\ln 3 < x\ln 2$$

$$\Rightarrow \quad x\ln 3 - x\ln 2 < 0 \quad \Rightarrow \quad x(\ln 3 - \ln 2) < 0$$

$$\Rightarrow \quad x\ln\tfrac{3}{2} < 0$$

And notice that this is true when $x < 0$. So in this region, we want the positive values: $-(3^x - 2^x)$.

So, $\quad A = \int_{-1}^{0}(-3^x + 2^x)dx + \int_{0}^{1}(3^x - 2^x)dx$

$$= \left[\tfrac{2^x}{\ln 2} - \tfrac{3^x}{\ln 3}\right]_{-1}^{0} + \left[\tfrac{3^x}{\ln 3} - \tfrac{2^x}{\ln 2}\right]_{0}^{1}$$

$$= \left[\left(\tfrac{1}{\ln 2} - \tfrac{1}{\ln 3}\right) - \left(\tfrac{1}{2\ln 2} - \tfrac{1}{3\ln 3}\right)\right] + \left[\left(\tfrac{3}{\ln 3} - \tfrac{2}{\ln 2}\right) - \left(\tfrac{1}{\ln 3} - \tfrac{1}{\ln 2}\right)\right]$$

$$= \left(\tfrac{1}{\ln 2} - \tfrac{1}{2\ln 2} - \tfrac{2}{\ln 2} + \tfrac{1}{\ln 2}\right) - \left(\tfrac{1}{\ln 3} + \tfrac{1}{3\ln 3} + \tfrac{3}{\ln 3} - \tfrac{1}{\ln 3}\right)$$

$$= \frac{2-1-4+2}{2\ln 2} + \frac{-3+1+9-3}{3\ln 3} = \frac{4}{3\ln 3} - \frac{1}{2\ln 2} \approx 0.4923.$$

It's the area between the two curves 3^x and 2^x: