# MATH 1271: Calculus I

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## 5.5 - Substitution Rule

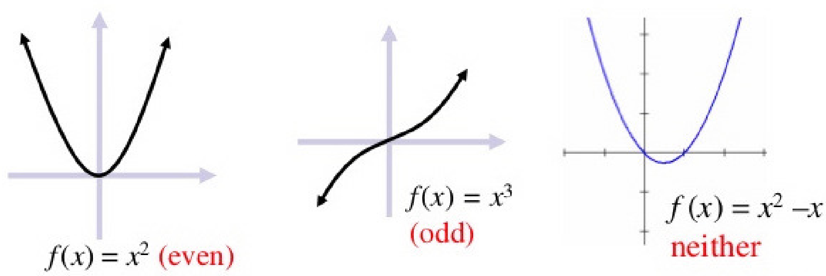
### Review:

**Substitution Rule**:
Recall that due to the chain rule we have: $(\sin(e^{2x}))' = \cos(e^{2x})(2e^{2x})$, or $(f(g))' = F(g) \cdot g'$. And since integrating is an opposite process from differentiating, we should be able to determine that $\int F(g) \cdot g' = f(g)$. So, let's say you had to integrate something of the form $\int F[g(x)] \cdot g'(x)dx$, or notated differently:
$\int F(u)du$, when $u = g(x)$. For example $\int \cos(e^{2x}) \cdot 2e^{2x}dx$. How could we go about solving this?

Notice that the expression $\int F(u)du$ is simpler than $\int F[g(x)] \cdot g'(x)dx$. But how did we eliminate $g'(x)$ and switch from $dx$ to $du$? Until now, we have treated the $dx$ as a notational appendage which merely told us which variable we were integrating over. The substitution rule works because it uses $dx$ operationally. Notice that if we take the derivative of $u = g(x)$ with respect to $x$, we get $\frac{du}{dx} = g'(x)$, or $du = g'(x)dx$. This allows us to make the necessary substitution above (try it!).

**Substitution with Definite Integrals**: $\int_{x=a}^{x=b} F(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} F(u)du$,
notice you have to change the bounds of integration!

**Integrals of Symmetric Functions**: If $F$ is continuous on $[-a, a]$, and...
- If $F$ is even, then $\int_{-a}^{a} F(x)dx = 2\int_{0}^{a} F(x)dx$.
- If $F$ is odd, then $\int_{-a}^{a} F(x)dx = 0$.




**Useful Trigonometric Integral**: $\int \tan x dx = \ln|\sec x| + C$.

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**Problem** 46. **Evaluate the Indefinite Integral**: $\int x^2\sqrt{2+x}\,dx$

Let $u = 2 + x$. Then $\frac{du}{dx} = 1$ or $du = dx$,

$x = u - 2$, so $x^2 = (u-2)^2$.

Therefore: $\int x^2 \sqrt{2+x}\, dx = \int (u-2)^2 \sqrt{u}\, du$

$= \int (u^2 - 4u + 4) u^{\frac{1}{2}}\, du = \int \left( u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}} \right) du$

$= \frac{2}{7} u^{\frac{7}{2}} - \frac{8}{5} u^{\frac{5}{2}} + \frac{8}{3} u^{\frac{3}{2}} + C.$ Are we done?

$= \frac{2}{7} (2+x)^{\frac{7}{2}} - \frac{8}{5} (2+x)^{\frac{5}{2}} + \frac{8}{3} (2+x)^{\frac{3}{2}} + C.$

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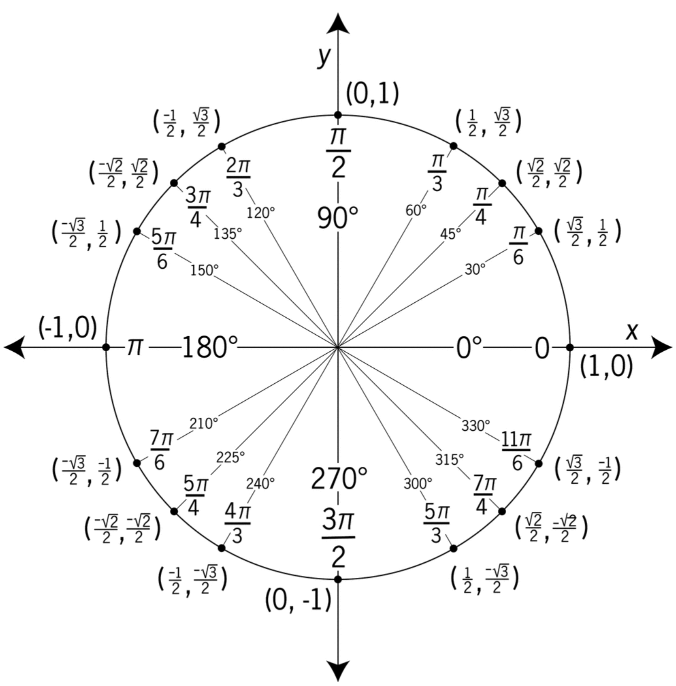
**Problem** 70. Evaluate the definite integral: $\int_{\frac{1}{2}}^{1} \frac{\cos^{-1} x}{-\sqrt{1-x^2}}\, dx.$

Notice the form $\frac{1}{-\sqrt{1-x^2}}$ is related to $(\cos^{-1} x)'$.

So, let: $u = \cos^{-1} x$, so $du = \frac{dx}{-\sqrt{1-x^2}}$. What else do I need to change?

When $x = \frac{1}{2}$, $u = \pm\frac{\pi}{3}$; when $x = 1$, $u = 0$.




Thus, $\int_{x=\frac{1}{2}}^{x=1} \frac{\cos^{-1} x}{-\sqrt{1-x^2}}\, dx = \int_{u=\pm\frac{\pi}{3}}^{u=0} u\ du$

$$= \left[\frac{u^2}{2}\right]_{\pm\frac{\pi}{3}}^{0} = \left(\frac{(0)^2}{2} - \frac{\left(\pm\frac{\pi}{3}\right)^2}{2}\right) = -\frac{\pi^2}{18}.$$

What didn't I need to change back to $x$ this time, and why?

Instead of changing the bounds of integration, we have the option of simply converting back to $x$ before applying the bounds:

$$\int_{x=\frac{1}{2}}^{x=1} \frac{\cos^{-1}x}{-\sqrt{1-x^2}}\,dx = \int_{x=\frac{1}{2}}^{x=1} u\ du = \left[\frac{u^2}{2}\right]_{x=\frac{1}{2}}^{x=1} = \left[\frac{(\cos^{-1}x)^2}{2}\right]_{x=\frac{1}{2}}^{x=1} = \left(\frac{(0)^2}{2} - \frac{\left(\pm\frac{\pi}{3}\right)^2}{2}\right) = -\frac{\pi^2}{18}.$$

Either way, we must still consult the unit circle!

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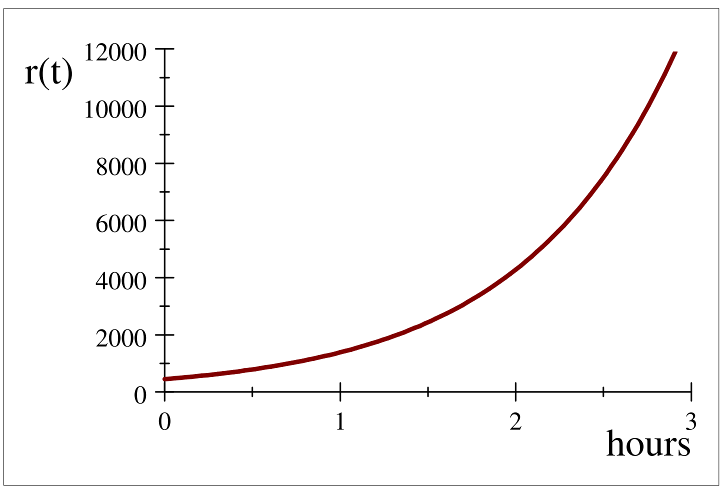
**Problem** 82. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour.

How many bacteria will there be after three hours?




So the growth rate is $r(t) = ae^{bt}$, with $a = 450.268$ and $b = 1.12567$.

Let $P(t)$ := population after $t$ hours.

Since $r(t) = P'(t)$…

$\int_0^3 r(t)dt = P(3) - P(0)$ is the total change in the population after 3 hours (net change theorem).

Since we start with 400 bacteria, the population at hour three will be…

$$P(3) = 400 + \int_0^3 r(t)dt = 400 + \int_0^3 ae^{bt}dt$$

$$= 400 + a\int_0^3 e^{bt}dt$$

$$= 400 + a\left[\tfrac{1}{b}e^{bt}\right]_0^3 = 400 + \tfrac{a}{b}(e^{3b} - e^{0\cdot b}) = 400 + \tfrac{a}{b}(e^{3b} - 1),$$

and substituting back in our constants:

$$P(3) = 400 + \tfrac{450.268}{1.12567}(e^{3(1.12567)} - 1) \approx 11,713 \text{ bacteria !?!}$$