MATH 1271: Calculus I

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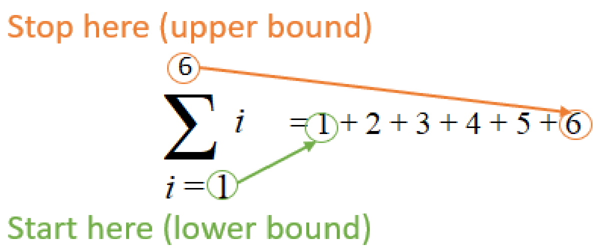
5.2 - Definite Integral

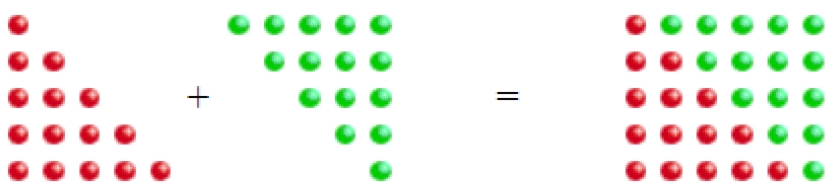
Review

Sigma Notation (Σ) **and Useful Sums**:



i is called the index

- $\sum_{i=1}^{n} i \;=\; 1 + 2 + 3 + \ldots + n \;=\; \frac{n(n+1)}{2}$,
- $\sum_{i=1}^{n} i^2 \;=\; 1^2 + 2^2 + 3^2 + \ldots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$,
- $\sum_{i=1}^{n} i^3 \;=\; 1^3 + 2^3 + 3^3 + \ldots + n^3 \;=\; \frac{n^2(n+1)^2}{4}$.



Visual proof that $2\sum_{i=1}^{n} i = n(n+1)$ or $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Some Obvious Properties

$\sum_{i=1}^{n} c = nc$	$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$	

Definite Integral:

$\int_a^b f(x)dx := \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

If the limit exists, the function f is (Riemann) integrable. **[Theorem 4]**

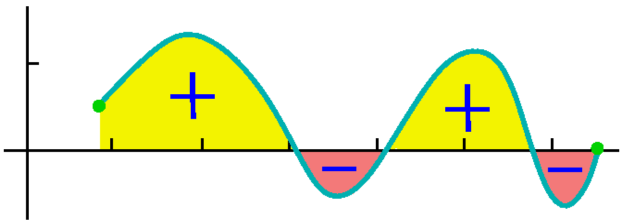
Right Hand Integration	Left Hand Integration
$\int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$	$\int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$

Midpoint Rule (**usually the best approximation**):

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(\bar{x}_i)\Delta x = [f(\bar{x}_1) + \ldots + f(\bar{x}_n)]\Delta x$$

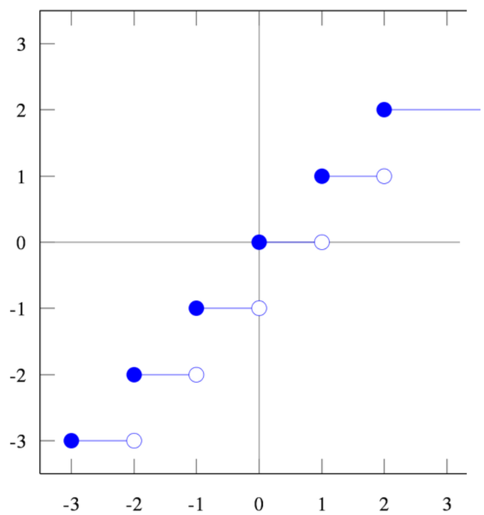
where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ = midpoint of the interval $[x_{i-1}, x_i]$.

So if integration measures the area between the curve and the x-axis, what happens when the curve dips below the axis? We get negative area! This introduces the idea of net area.
Net Area: If f takes on both positive and negative values, the integral represents the **net area**, that is, the area above the curve minus the area below the curve.



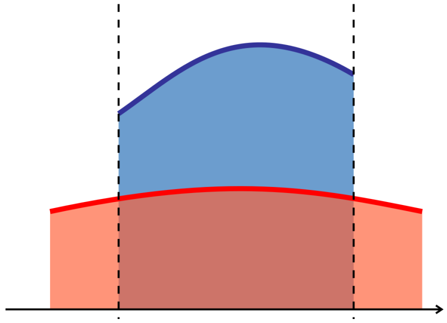
What types of functions can we integrate?

Existence of Definite Integral: If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$. Recall that $\int_a^b f(x)dx$ is defined as a limit of a sum of rectangles. So, this theorem says that if the conditions above are met, that limit exists (in this context, "exists" means that $\int_a^b f(x)dx$ is equal to a non-infinite real number).

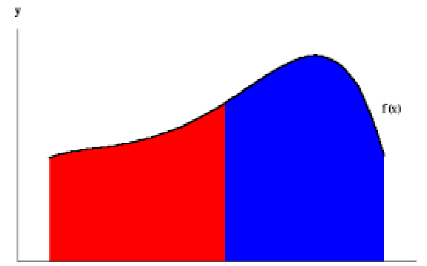


Properties of Integrals: Let c be any constant, then:

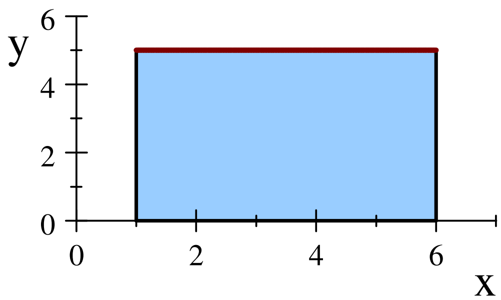
- $\int_a^b f\,dx = -\int_b^a f\,dx,$
- $\int_a^a f\,dx = 0,$
- $\int_a^b (f-g)dx = \int_a^b f\,dx - \int_a^b g\,dx,$
- $\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx,$
- $\int_a^b c\,dx = c\left(b-a\right),$
- $\int_a^b (c \cdot f(x))dx = c\int_a^b f(x)\,dx.$



$$\int_a^b (f-g)dx = \int_a^b f\,dx - \int_a^b g\,dx$$

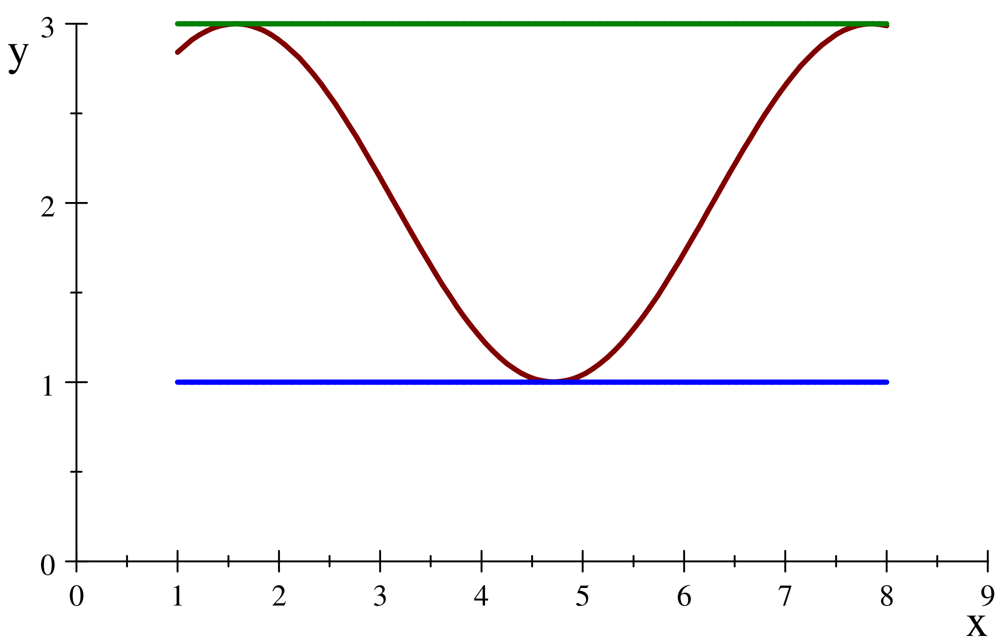


$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$$



$$\int_1^6 5dx = 5\left(6-1\right) = 25$$

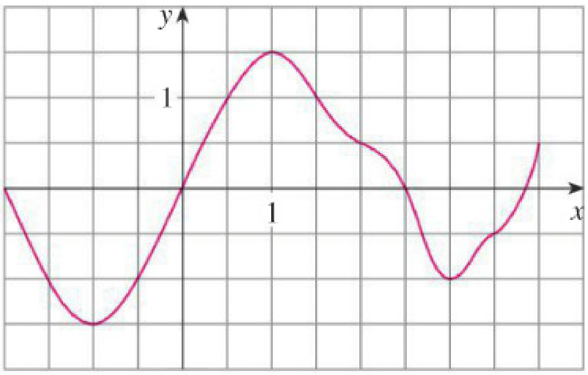
- ♦ if $f \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
- ♦ if $f \geq g$ for $a \leq x \leq b$, then $\int_a^b f\,dx \geq \int_a^b g\,dx$.
- ♦ if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$ where $m, M \in \mathbb{R}$.



If $1 \leq \sin x + 2 \leq 3$, then $1(8-1) \leq \int_1^8 (\sin x + 2)dx \leq 3(8-1)$

Problem 6. The graph of g is shown. Estimate $\int_{-2}^{4} g(x)dx$ with six sub-intervals using:

(a) right endpoints, (b) left endpoints, and (c) midpoints.



Right Endpoints: $\int_{-2}^{4} g(x)dx \approx \left[g(-1) + g(0) + \ldots + g(4)\right]\Delta x$

$= (-1.5 + 0 + 1.5 + 0.5 - 1 + 0.5)(1) = 0.$

Left Endpoints: $\int_{-2}^{4} g(x)dx \approx \left[g(-2) + g(-1) + \ldots + g(3)\right]\Delta x$

$= (0 - 1.5 + 0 + 1.5 + 0.5 - 1)(1) = -\frac{1}{2}.$

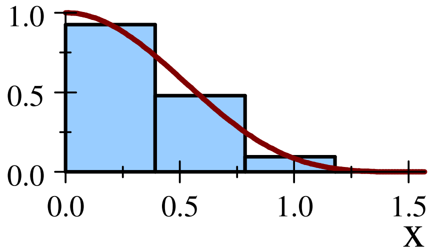
Midpoints: Home Exercise!

Problem 10. Use the midpoint rule with $n = 4$ to approximate the integral $\int_{0}^{\frac{\pi}{2}} \cos^4 x dx$. Round the answer to four decimal places.

$\Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}.$

$\int_{0}^{\frac{\pi}{2}} \cos^4 x dx$

$\approx [\cos^4(\frac{\pi}{16}) + \cos^4(\frac{3\pi}{16}) + \cos^4(\frac{5\pi}{16}) + \cos^4(\frac{7\pi}{16})]\frac{\pi}{8} \approx 0.5890.$



$\cos^4 x$

Problem 18. Express the limit as a definite integral on the given interval.

$\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i} \Delta x_i, \qquad [\pi, 2\pi]$

$\int_{\pi}^{2\pi} \frac{\cos x}{x} dx.$

Problem 24. Use the form of the definition of the integral given in Theorem 4

to evaluate the integral: $\int_0^2 (2x - x^3) dx.$

Recall Theorem 4: if f is integrable, then $\int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Note that $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0 + i\Delta x = \frac{2i}{n}$.

So, $\int_0^2 (2x - x^3) dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(\frac{2i}{n}) \frac{2}{n}$

$= \lim\limits_{n\to\infty} \sum_{i=1}^{n} \left[2(\frac{2i}{n}) - (\frac{2i}{n})^3 \right] \frac{2}{n}$

What do we do with this? Recall the "useful sums," and properties of sigmas.

$= \lim\limits_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8i}{n^2} - \frac{16i^3}{n^4} \right]$

$= \lim\limits_{n\to\infty} \left[\sum_{i=1}^{n} \frac{8i}{n^2} - \sum_{i=1}^{n} \frac{16i^3}{n^4} \right]$

$$= \lim_{n\to\infty} \left[\frac{8}{n^2} \sum_{i=1}^{n} i - \frac{16}{n^4} \sum_{i=1}^{n} i^3 \right]$$

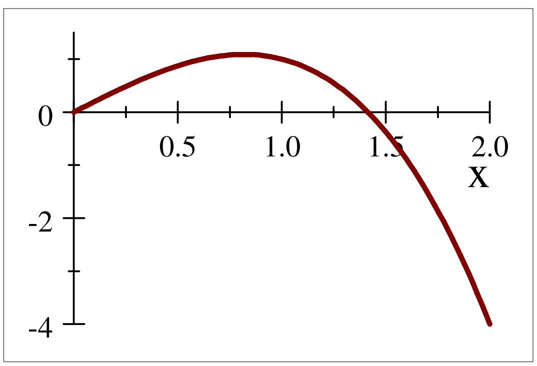
Recall you can pull a constant out of a sum, and to the sum, the *n* is just a constant. Yes, the limit is changing *n*, but for each sum, it is just a constant.

$$= \lim_{n\to\infty} \left[\frac{8}{n^2} \frac{n(n+1)}{2} - \frac{16}{n^4} \frac{n^2(n+1)^2}{4} \right]$$

$$= \lim_{n\to\infty} \left[4\frac{n^2+n}{n^2} - 4\frac{(n+1)^2}{n^2} \right]$$

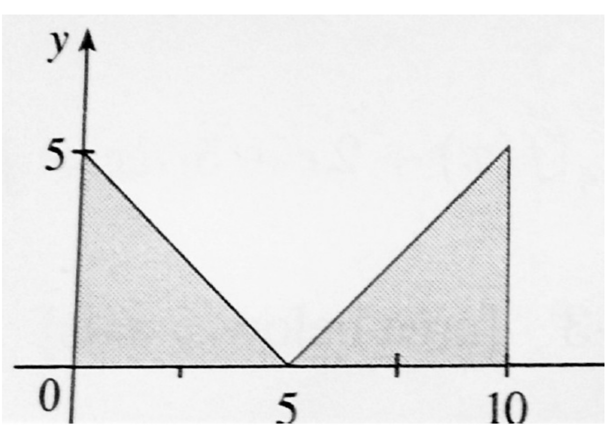
$$= \lim_{n\to\infty} \left[4(1+\tfrac{1}{n}) - 4(1+\tfrac{1}{n})^2 \right]$$

$$= 4 \cdot 1 - 4 \cdot 1 = 0.$$



$2x - x^3$

Problem 40. Evaluate the integral $\int_0^{10} |x-5| dx$ by interpreting it in terms of areas.



This function can be interpreted as the sum of the areas of the 2 shaded triangles;

that is, $2 \cdot \left(\text{Area of Triangle}\right) = 2(\frac{1}{2})(\text{width})(\text{height}) = 2(\frac{1}{2})(5)(5) = 25$ units.