MATH 1271: Calculus I

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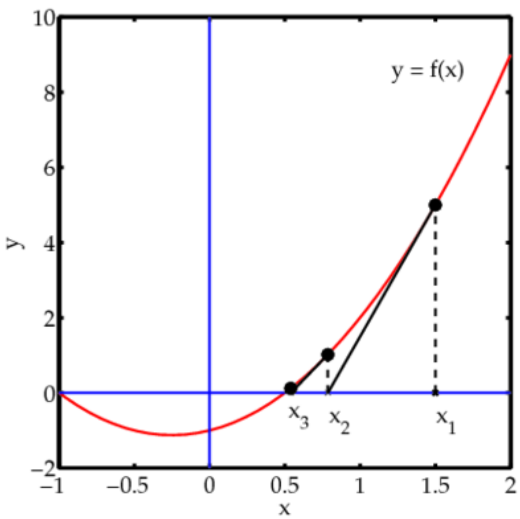
4.8 - Newton's Method

Review:



Newton's Method: Finding roots for an equation of the form $f(x) = 0$.

1. Make a guess (x_1) (an initial approximation), where you believe one of the roots might be near.
   (possibly by using the curve sketching methods learned earlier).

2. Calculate the derivative of the function at that point $f'(x_1)$.

3. Since a derivative defines a slope, we can write down the equation of the tangent line (using point-slope form): $y - f(x_1) = f'(x_1)(x - x_1)$.

4. We are interested in where this line intersects the x-axis (a root), which would be at: $(x,y) = (x_2, 0)$. Plugging $(x_2, 0)$ into our tangent line EQ, we get:
   $0 - f(x_1) = f'(x_1)(x_2 - x_1)$.

5. Solve for x_2 (a new approximation for our root). We get $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

6. Repeat steps 2 thru 5 until you reach the desired accuracy.
   x_i is considered accurate if $f(x_i)$ is sufficiently close to zero for your needs.

More generally, we have (from step 5): $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

If the numbers x_n become closer and closer to a particular number r (that is, if they converge to a "limiting value"), we say $\lim_{n\to\infty} x_n = r$. This is our goal!

If this process fails, it is probably due to an unfortunate guess in step 1) for x_1. Attempt another guess, and try again.

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Problem 16. Use Newton's method to approximate the positive root of $3\sin x = x$, correct to six decimal places.

Looking at the graph, we estimate a root near $x_1 = 2$.

Let $f(x) = 3\sin x - x$.

Want to use tangent line at $\left(x_1, f(x_1)\right)$ (which is $y - f(x_1) = f'(x_1)(x - x_1)$) to discover a new estimate x_2 at the x-intercept of this tangent line.

$f'(x) = 3\cos x - 1$

Specifically for $x_1 = 2$,

$f'(x_1) = f'(2) = 3\cos 2 - 1 \approx -2.24844051.$

$f(x_1) = f(2) = 3\sin 2 - 2 \approx 0.72789228.$

So, plugging this into our tangent line: $y - 0.72789228 = -2.24844051(x-2)$.

And then locating x_2, which is the x-intercept, $(x,y) = (x_2, 0)$ of the tangent line:

$0 - 0.72789228 = -2.24844051(x_2 - 2) = 4.49688102 - 2.24844051x_2$.

So, $2.24844051x_2 \approx 0.72789228 + 4.49688102 = 5.2247733$, and $x_2 \approx \frac{5.2247733}{2.24844051} = 2.323732061$.

You can also use the less intuitive, but more efficient equation:
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3\sin x_n - x_n}{3\cos x_n - 1}$.

So, $x_2 = x_1 - \frac{3\sin x_1 - x_1}{3\cos x_1 - 1} = 2 - \frac{3\sin 2 - 2}{3\cos 2 - 1} \approx 2.323732061$.

$x_3 = 2.323732 - \frac{3\sin(2.323732) - (2.323732)}{3\cos(2.323732) - 1} \approx 2.2795948197$.

$x_4 \approx 2.278863$. $\quad x_5 \approx x_4$.

So the positive root is 2.278863, to six decimal places.

And note that $f(x_4) = 3\sin(2.278863) - 2.278863 \approx -0.0000010031$,
which is fairly close to zero!

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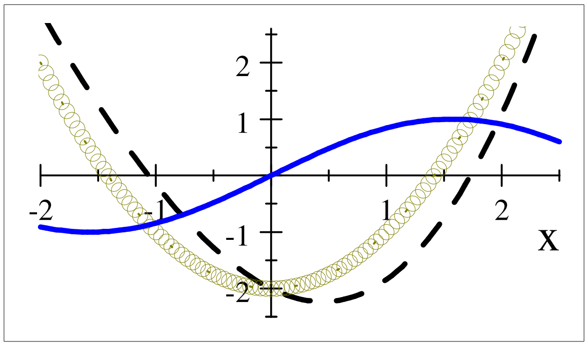
Problem 22. Use Newton's method to find all roots of $\sin x = x^2 - 2$ correct to six decimal places.

Graphs of $\sin x$ (solid), $x^2 - 2$ (hollow circles), and $x^2 - 2 - \sin x$ (dashed).



From the graph (dashed), we see that there appear to be points of intersection near $x = -1$ and $x = 2$. Finding the roots for $\sin x = x^2 - 2$ is the same as solving $f(x) = -\sin x + x^2 - 2 = 0$.

$$f'(x) = -\cos x + 2x, \text{ so } \ldots$$

$$x_{n+1} = x_n - \frac{\sin x_n - x_n^2 + 2}{\cos x_n - 2x_n}.$$

Substituting are approximations of the root into the previous equation repeatedly, we find:

$x_1 = -1$	$x_1 = 2$
$x_2 \approx -1.062406$	$x_2 \approx 1.753019$
$x_3 \approx -1.061550 \approx x_4$	$x_3 \approx 1.728710$
	$x_4 \approx 1.728466 \approx x_5.$

To 6 decimal places, the roots of the equation are -1.061550 and 1.728466.