MATH 1271: Calculus I

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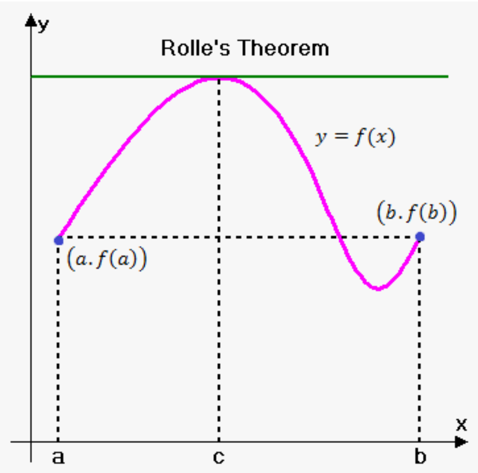
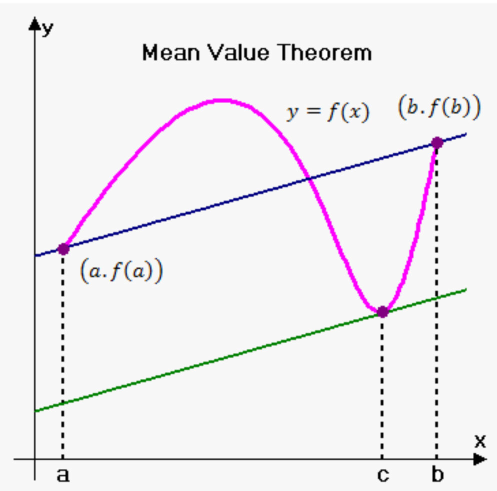
4.2 - The Mean Value Theorem

Review:



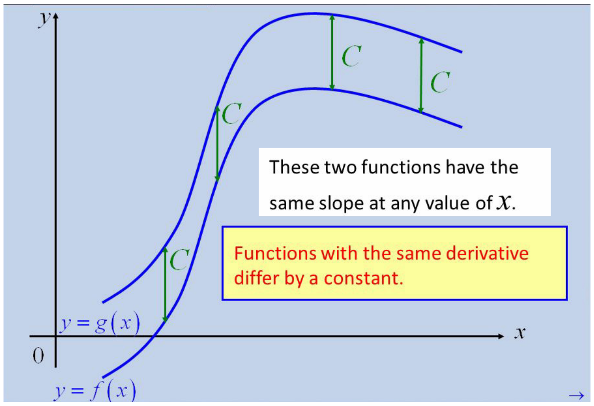
Assuming f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, we have:
Rolle's Theorem: There is a number c in (a,b) such that: $f'(c) = 0$.



A more general version of Rolle's theorem is the Mean Value Theorem.
Assuming f is continuous on $[a,b]$, and differentiable on (a,b), then we have:
Mean Value Theorem: There is a number c in (a,b) such that: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Constant Interval Theorem: If f is continuous and $f'(x) = 0$ for all x in an interval (a,b), then f is constant on (a,b).

Equal Derivative Corollary: If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f - g$ is constant on (a,b); that is, $f(x) = g(x) + C$ where C is a constant.



Problem 4. Verify that $f(x) = \cos 2x$ satisfies the three hypotheses of Rolle's theorem on the interval $[\frac{\pi}{8}, \frac{7\pi}{8}]$. Then find all numbers c that satisfy the conclusion of Rolle's theorem.

f, being the composite of two other differentiable functions (the cosine function and the polynomial $2x$), is continuous and differentiable on all of $\mathbb{R}$, so it is certainly continuous on $\left[\frac{\pi}{8}, \frac{7\pi}{8}\right]$ and differentiable on $\left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$. Also, $f(\frac{\pi}{8}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{7\pi}{4} = f(\frac{7\pi}{8})$.

So now we need to find a c such that: $f'(c) = 0$

$\Rightarrow \quad -2\sin 2c = 0$

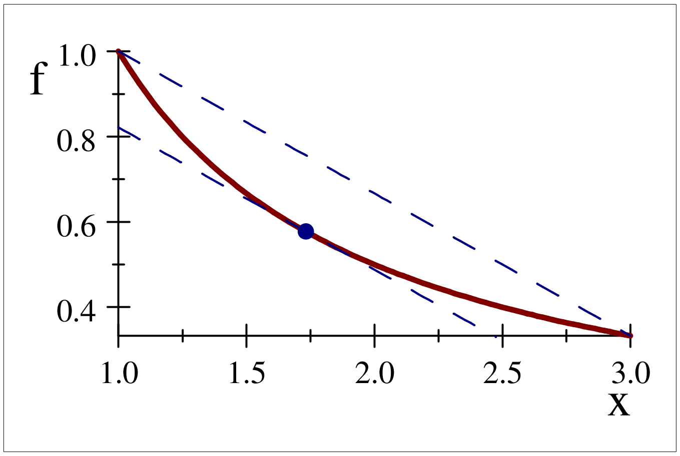
$\Rightarrow \quad \sin 2c = 0$

$\Rightarrow \quad 2c = n\pi$, where n is any integer (because any multiple of π gives us $\sin(n\pi) = 0$)

So, $c = n\frac{\pi}{2}$.

However, they are not all in our interval!

If $n = 0$, then $c = 0 < \frac{\pi}{8}$. If $n = 2$, then $c = \frac{2\pi}{2} = \pi > \frac{7\pi}{8}$.
However, if $n = 1$, then $c = \frac{\pi}{2} = \frac{4\pi}{8}$, which is in the open interval $\left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$, so $c = \frac{\pi}{2}$ is the only point from $n\frac{\pi}{2}$ that verifies the conclusion of Rolle's Theorem.



$\cos 2x$

Problem 12. Verify that $f(x) = \frac{1}{x}$ satisfies the hypotheses of the mean value theorem on the interval $[1,3]$. Then find all numbers c that satisfy the conclusion of the mean value theorem.

Notice that f is continuous on $(-\infty,0) \cup (0,\infty)$ (AKA, everywhere except 0).

Then, observe that $f' = -\frac{1}{x^2}$. So f' is also differentiable on $(-\infty,0) \cup (0,\infty)$. So f is certainly continuous on $[1,3]$ and differentiable on $(1,3)$.

So now we need to find a c such that $f'(c)$ is equal to:

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{\frac{1}{3}-1}{2} = -\frac{1}{3}$$, the slope of the secant line.

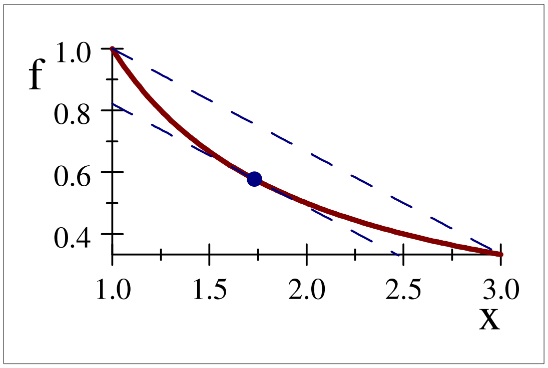
Taking the derivative:

$$f'(c) = -\frac{1}{c^2}$$

$$\Rightarrow \quad -\frac{1}{c^2} = -\frac{1}{3}$$

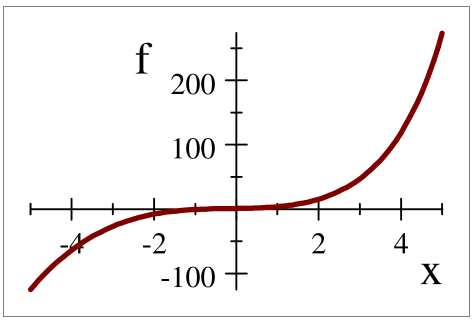
$$\Rightarrow \quad c = \pm\sqrt{3}.$$

But only $+\sqrt{3} \approx 1.732$ is in $(1,3)$.



$\frac{1}{x}$

Problem 18. Show that $f(x) = x^3 + e^x$ has **exactly** one real root.



First let's use IVT to show that it has at least one root.

$f(-1) = -1 + \frac{1}{e} < 0$ and $f(0) = 1 > 0$.
(Pro-Tip: if you have to try plugging in some numbers, start out with $0, 1$. They are generally easy to plug-in!)

Since $x^3 + e^x$ is the sum of a polynomial and the natural exponential function (both of which are differentiable for all x), then f is continuous and differentiable for all x. By IVT, there is a number c in $(-1, 0)$ such that $f(c) = 0$. Thus, the given equation has **at least** one real root, but are there more?

If we make the dubious assumption that $x^3 + e^x$ has **two** (or more) distinct real roots (which means roots a and b with $a < b$), then we have $f(a) = f(b) = 0$. However, since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's theorem tells us that there is a number d in (a, b) such that $f'(d) = 3d^2 + e^d = 0$.

However, observe that $3d^2 + e^d$ is always greater than zero, never equal to it.
(put another way, can e^d ever equal $-3d^2$? No!!)

This contradiction shows that our dubious assumption was incorrect, and that the given equation can't have two (or more) distinct roots, so it must have exactly one root. ■

Problem 26. Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [Hint: Apply the mean value theorem to the function $h = f - g$.]

Let $h = f - g$.

Notice that in terms of h, we are trying to prove that $h(b) < 0$.

Then, since f and g are continuous on $[a,b]$ and differentiable on (a,b), so is h. And thus h satisfies the assumptions of the mean value theorem.

Therefore, there is a number c with $a < c < b$ such that $h'(c) = \frac{h(b)-h(a)}{b-a}$.

Multiplying by $(b-a)$, we have :
$h'(c)(b-a) = h(b) - h(a) = h(b) - [f(a) - g(a)] = h(b) - 0 = h(b).$ $(*)$
(notice I used the fact given to us that $f(a) = g(a)$).

So again, our goal (incorporating this previous equation) is to show that: $h(b) = h'(c)(b-a) < 0$.

However, notice that $h'(c) = f'(c) - g'(c) < 0$ (I used the fact given to us that $f'(x) < g'(x)$ for all x).

And also notice that $b - a > 0$.

Therefore, we have $h(b) = h'(c)(b-a) < 0$.

Recall that $h(x) = f(x) - g(x)$.

So, $f(b) - g(b) = h(b) < 0$, and hence $f(b) < g(b)$. ■