MATH 1271: Calculus I

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3.9 - Related Rates

Review

Related Rates: Imagine a situation where you want to compute the rate-of-change (derivative) of one quantity (for example, volume V). But you find it difficult! However, what if you DO know the rates-of-change of some more easily measured quanties involved in your situation (for example, height and width: h, w). So, to find the volume derivative, you find an equation that relates the quantities (e.g, $V = 10hw$), then differentiate both sides with respect to time (using the chain rule), and solve for the derivative you seek.

Problem Solving Strategies

♦ Read the problem carefully.
♦ Draw a diagram if possible.
♦ Introduce notation. Assign symbols to all quantities that are functions of time (for example V, h, w).
♦ Express the given information and the required rate in terms of derivatives (for example: Aassume we know that $h' = 5$ and want to know V' when $h = 3$).
♦ Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (for example, assume we know: $V = 10hw$ and $w = 2h$, so $V = 20h^2$).
♦ Use the chain rule to differentiate both sides of the equation with respect to t (for example $V' = 40hh'$).
♦ Substitute the given information into the resulting equation and solve for the unknown rate (for example at $h = 3$ we have: $V' = 40(5)(3) = 600$).

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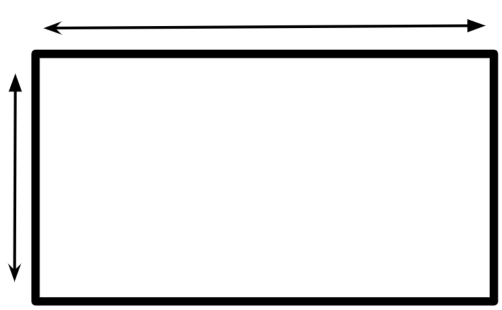
Problem 4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

We might draw the graph:



$A = lw, \qquad \frac{dl}{dt} = 8, \qquad \frac{dw}{dt} = 3.$

$\Rightarrow \frac{dA}{dt} = \;??$

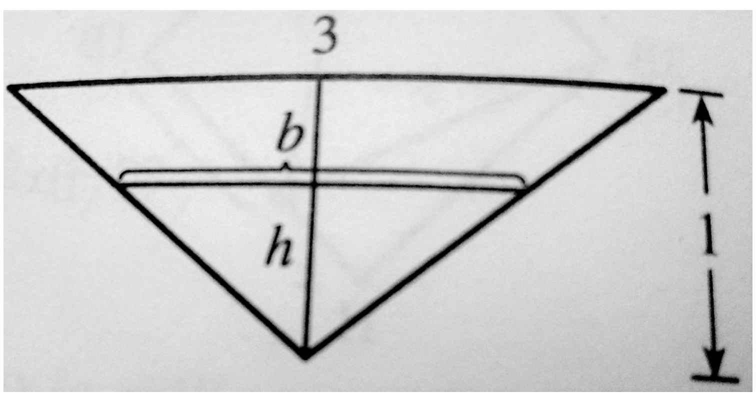
$= l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}$ (product rule)

$= l(3) + w(8) = 3l + 8w.$ When length is 20 cm and width is 10 cm?

$= (3)20 + (8)10 = 140 \ \frac{cm^2}{s}.$

Problem 24. A trough is 10 *ft* long and its cross-section has the shape of an isosceles triangle that is 3 feet across at the top and has a height of 1 foot. If the trough is being filled with water at a rate of 12 $\mathrm{ft}^3/\mathrm{min}$, how fast is the water level rising when the water is 6 inches deep?

We might draw the graph:



We wish to find $\frac{dh}{dt}$. But what quantity can we calculate with what we are given?

Notice we are given details necessary to calculate the volume $V = \frac{1}{2}bhl$ of water in the trough.

But that is a lot of letters! How can we eliminate some of them?

$V = \frac{1}{2}bh(10)$ (we were given the length)

Recall: "Two triangles are said to be similar if their corresponding angles are congruent and the corresponding sides are in proportion." Sufficient if you have AAA, SSS, or SAS. In this case, we have AAA.

By similar triangles (the full cross-section of the trough, and the partially filled cross-section):

$\frac{3}{1} = \frac{b}{h}$, so $b = 3h$. So, the trough has volume…

$V = 5(3h)h = 15h^2$

Recall, $\frac{dV}{dt} = 12$.

So, $12 = 30h\frac{dh}{dt}$.

Or, $\frac{dh}{dt} = \frac{12}{30h} = \frac{2}{5h}$ "When the water is 6 inches deep?"

$= \frac{2}{5 \cdot \frac{1}{2}} = \frac{4}{5}\ ft/\min$.

Problem 34. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is some constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 $kPa/\min$. At what rate is the volume increasing at this instant?

We are given an equation, and asked about a rate of change $\frac{dV}{dt}$, so we probably want to take the derivative of the equation.

$P'V^{1.4} + P(1.4V^{0.4}V') = 0$

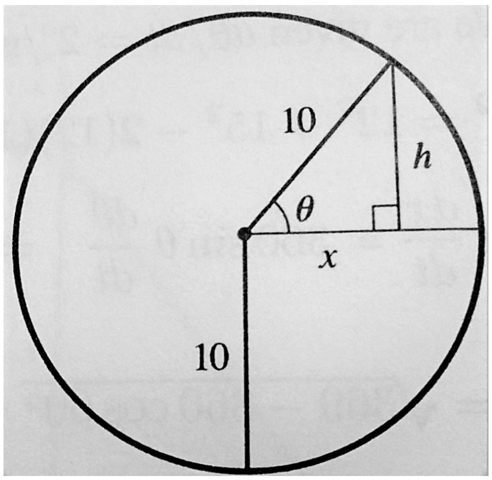
$\Rightarrow V' = -\frac{V^{1.4}}{1.4PV^{0.4}}P' = -\frac{V}{1.4P}P'$. (solving for $\frac{dV}{dt}$)

When $V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$.

Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36\ \frac{cm^3}{\min}$.

Problem 42. A Ferris wheel with a radius of 10 *m* is rotating at a rate of one revolution every two minutes. How fast is a rider rising when her seat is 16 m above ground level?

We might draw the graph:



We are given that $\frac{d\theta}{dt} = \frac{2\pi \ rad}{2 \ \min} = \pi \ rad/\min$.

We wish to use some equation involving the relavant quantities: angle θ, the radius 10, and the information sought regarding h. And since the information requested is a derivative, we will probably wish to take the derivative of that equation. So what equation connects those 3 bits of information?

From the figure, $\sin\theta = \frac{h}{10} \Rightarrow h = 10 \sin\theta$.

So $\frac{dh}{dt} = 10 \cos\theta \frac{d\theta}{dt}$

$h' = 10\pi \cos\theta$.

So by Soh-Cah-Toa we have $\cos\theta = \frac{adj}{hyp} = \frac{x}{10}$.

By the Pythagorean theorem, when $h = 6$ and $r = 10$, we have $x = \sqrt{100 - 36} = 8$.

Therefore, $\frac{dh}{dt} = 10(\frac{8}{10})\pi = 8\pi \ m/\min$.