MATH 1271: Calculus I

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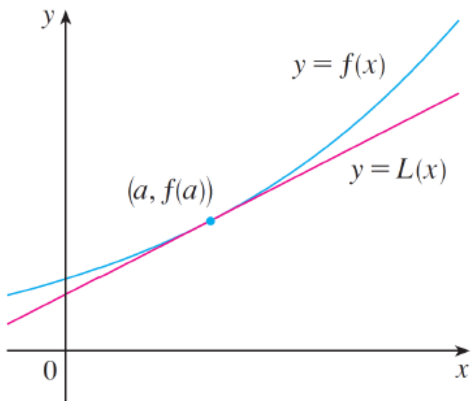
3.10 - Linear Approximations and Differentials

Review

Linearization of f at a: The linear function whose graph is the tangent line,
that is, $L(x) = f(a) + f'(a)(x - a)$.
So, $L(x)$ = initial height + $\frac{rise}{run} \cdot run = f(x)$ approximation.



Differentials:
Consider $y = f(x)$, where f is a differentiable function.
We know how to calculate the derivative $f'(x)$. And if we look at the tangent line starting at (x_0, y_0) on the graph of $f(x)$, we can calculate the slope $f'(x_0)$. We can now can define a NEW function dy, written like $dy = f'(x_0)dx$ (this should remind you of the equation for a line $y = mx$). If we were to travel along the x-axis by an amount equal to dx, then dy is the change in the height of the tangent line starting at (x_0, y_0). Note that since we are traveling along the tangent line, and not the function itself, dy will most likely be a different value than Δy (which is the symbol representing the change in the ACTUAL function when traveling dx or Δx, which both represent the same thing). dx and dy are called **differentials**, they are approximations based on a derivative.

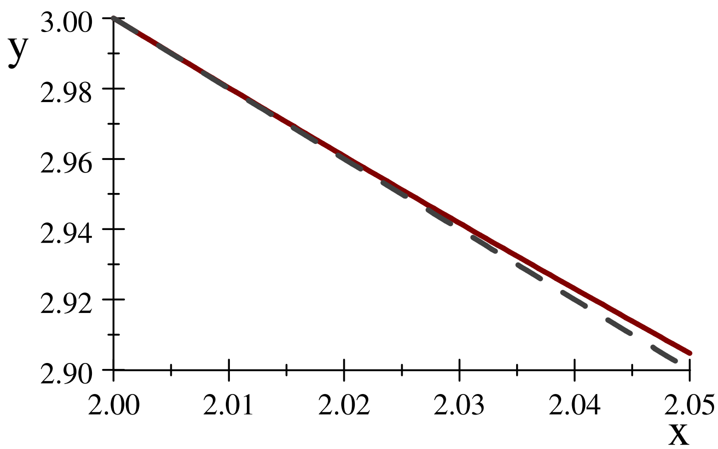
Stated another way: $dx = \Delta x$ represents the size of the interval over which you are considering the linearization of f. While $dy = \Delta L$ represents the size of the change of the linearization L of f over dx.

Problem 18. Find the differential dy and evaluate it for the given values of x and dx.

$y = \frac{x+1}{x-1}, \qquad x_0 = 2, \qquad dx = 0.05$

$$dy = f'(x_0)dx = \left(\frac{(1)(x-1)-(x+1)(1)}{(x-1)^2}\Big|_{x=2}\right)dx = -2dx$$

$dy = -2(0.05) = -0.1.$



$y = \frac{x+1}{x-1}$. Dashed line is the Linearization

Problem 22. Compute Δy and dy for $y = e^x$ with $x_0 = 0$ and $dx = \Delta x = 0.5$. Then sketch a diagram showing the line segments with lengths $dx, dy,$ and Δy.

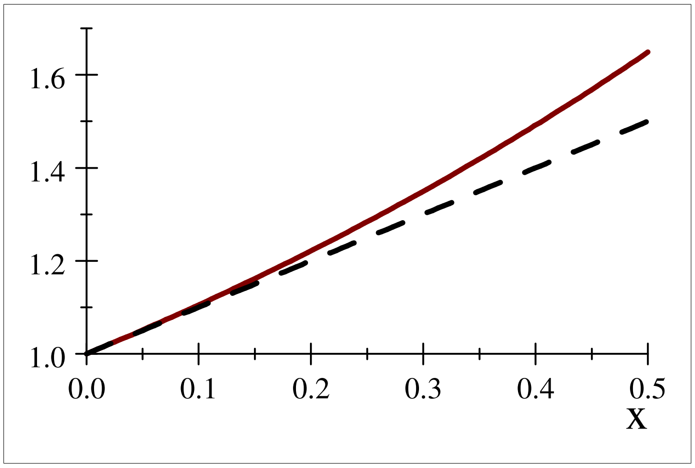
$\Delta y = f(0.5) - f(0)$

$= \sqrt{e} - 1 \approx 0.65.$

$dy = \ldots$

$= e^x dx$

$= e^0(0.5) = 0.5.$



$y = e^x$

Problem 26. Use a linear approximation (differentials) to estimate the value for $\frac{1}{4.002}$. (Imagine you didn't have a calculator. What function does this number resemble?)

$y = \frac{1}{x}$ near $x = 4$.

$\Rightarrow dy = -\frac{1}{x^2}dx.$

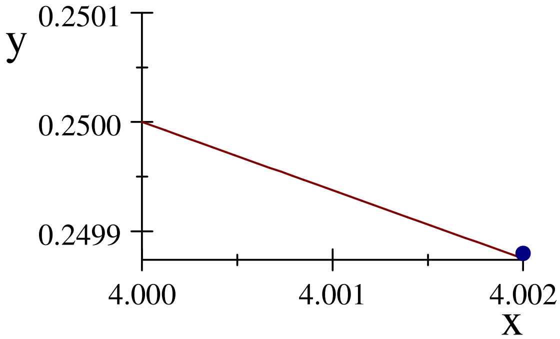
When $x = 4$ and $dx = 0.002$,

$dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so

$\frac{1}{4.002} \approx f(4) + dy$

$= \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875.$ ■

Which is off by about .001%



$\frac{1}{x}$. Black dot is the approximation

Problem 30. Explain, in terms of linear approximations or differentials, why the approximation $(1.01)^6 \approx 1.06$ is reasonable.

If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope $y' = 6(1)^5 = 6$.

And if the change in x is 0.01, then the change in y on the tangent line is $dy = 6(0.01) = 0.06$.

So approximating $(1.01)^6$ with 1.06 is reasonable.