

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

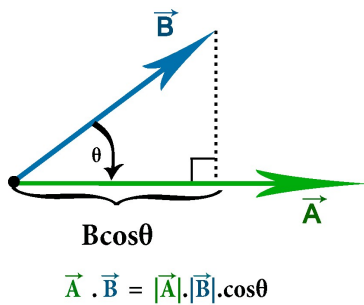
12.3 - Dot Product

Dot Product: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \vec{a} and \vec{b} , notated $\vec{a} \cdot \vec{b}$ or $\langle \vec{a}, \vec{b} \rangle$, is given by $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. Observe that with this type of "multiplication," we start with two vectors, but end up with a scalar. As a result, this is also called the **scalar product**, or inner product. In the next section, we will learn of a different type of vector multiplication where the result is a vector (vector product or cross product).

Properties of the Dot Product

- ◆ $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = (\sqrt{a_1^2 + \dots + a_n^2})^2 = a_1^2 + \dots + a_n^2$, (dot product of a vector with itself is the square of its magnitude).
- ◆ $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ Commutativity. ◆ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ Distributivity.
- ◆ $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$ Scalar Associativity. ◆ $\vec{0} \cdot \vec{a} = 0$ Scalar Zero.

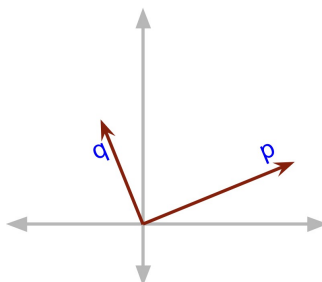
Dot Product Angle Theorem: If θ is the angle between \vec{a} and \vec{b} where $0 \leq \theta \leq \pi$, then $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.



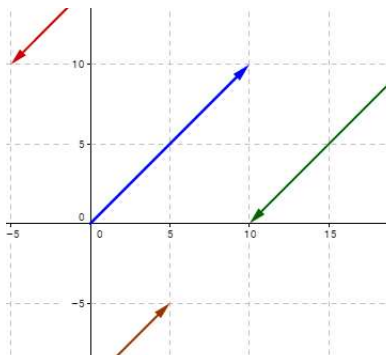
Corollary: Let θ be the angle between the (nonzero) vectors \vec{a} and \vec{b} , then: $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$.

Perpendicular/Orthogonal: Say we have nonzero vectors \vec{a} and \vec{b} where the angle between them is $\theta = \frac{\pi}{2}$. In this case, the previous theorem gives us $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \frac{\pi}{2} = 0$. Conversely, if $\vec{a} \cdot \vec{b} = 0$, then $\cos \theta = 0$ and the angle θ must be equal to $\frac{\pi}{2}$. The zero vector $\vec{0}$ is considered to be perpendicular to all vectors, therefore we have:

Vector Orthogonality Criteria: Two vectors \vec{p} and \vec{q} are orthogonal if and only if $\vec{p} \cdot \vec{q} = 0$. (see <https://www.geogebra.org/m/c8euxmbc>)



Parallel: Also observe that $\cos \theta = \pm 1$ when \vec{a} and \vec{b} are parallel to each other. Therefore, we can determine whether two vectors are parallel by looking at whether $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \pm 1$.

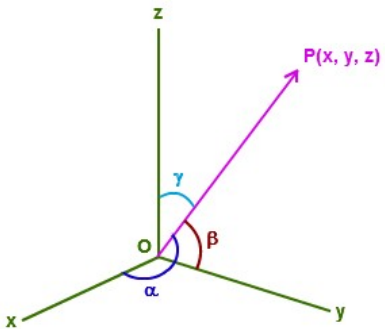


Useful Tidbit: You can easily experiment to find that $\vec{a} \cdot \vec{b}$ is positive for $\theta < \frac{\pi}{2}$, and negative for $\theta > \frac{\pi}{2}$. We can therefore think of $\vec{a} \cdot \vec{b}$ as measuring the extent to which the two vectors point in the same direction: positive when in the same direction, 0 when perpendicular, and negative when pointing in the opposite direction.

Direction Angles and Direction Cosines

Let \vec{P} be a nonzero vector in three dimensions.

Direction Angles: The angles α , β , and γ are those angles (all in the interval $[0, \pi]$) that the vector \vec{P} makes with the positive x , y , and z -axes, respectively.



Direction Cosines: $\cos \alpha$, $\cos \beta$, $\cos \gamma$.

Simple calculations reveal that using these angles, we can rewrite \vec{P} as:

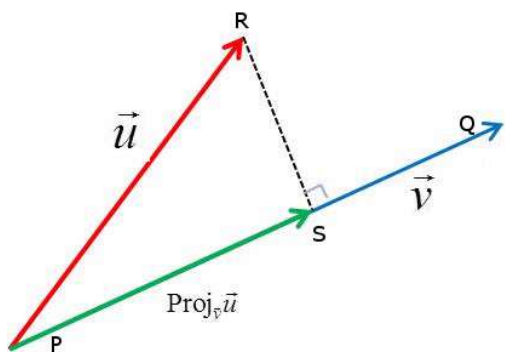
$$\vec{P} = \langle x, y, z \rangle = \langle |\vec{P}| \cos \alpha, |\vec{P}| \cos \beta, |\vec{P}| \cos \gamma \rangle = |\vec{P}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle.$$

Therefore, the unit vector is: $\frac{\vec{P}}{|\vec{P}|} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$.

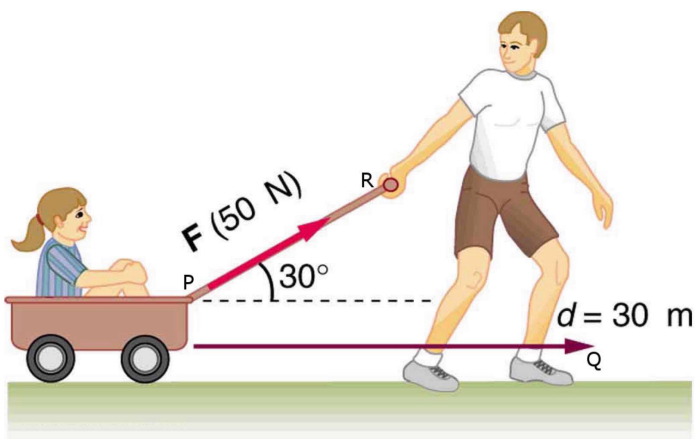
Projections

Vector Projection: (see image below) If point S (on \vec{v}) is the foot of the perpendicular from tip R of \vec{u} onto \vec{v} (which has tail P), then the vector with representation \vec{PS} is called the vector projection of \vec{u} onto \vec{v} and is denoted by $proj_{\vec{v}} \vec{u}$ (also thought of as the "shadow" of \vec{u}). So we have:

Vector Projection of \vec{u} onto \vec{v} : $proj_{\vec{v}} \vec{u} = \left(\frac{\vec{v} \cdot \vec{u}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|^2} \vec{v}.$



Scalar Projection of \vec{u} onto \vec{v} : This is the signed magnitude (scalar component) of the vector projection, which is the number $\frac{\vec{v} \cdot \vec{u}}{|\vec{v}|} = |\vec{u}| \cos \theta$, where θ is the angle between the vectors, denoted $\text{comp}_{\vec{v}} \vec{u}$. So we have **Scalar Projection of \vec{u} onto \vec{v} :** $\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|}$,



Displacement Vector: Suppose that some continuing force is written as a vector $\vec{F} = \overrightarrow{PR}$. And suppose this force is moving an object from P , but in a direction not parallel to \overrightarrow{PR} (see image above where the object is moved along the ground, instead of in the direction of the force). If the force moves the object from P to Q , then the displacement vector is notated $\vec{D} := \overrightarrow{PQ}$.

Work: This is the product of the distance moved ($|\vec{D}|$) by the the component of force along \vec{D} (i.e., $|\vec{F}| \cos \theta$). So we have: $W = |\vec{D}| (|\vec{F}| \cos \theta)$. Therefore, using the dot product angle theorem, we have $W = |\vec{F}| |\vec{D}| \cos \theta = \vec{F} \cdot \vec{D}$.

Problem #4 Letting $\vec{a} = \langle 6, -2, 3 \rangle$ and $\vec{b} = \langle 2, 5, -1 \rangle$, find $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = 6 \cdot 2 + (-2) \cdot 5 + 3 \cdot (-1) = -1.$$

Problem #18 Find the angle between the vectors $\vec{a} = \langle 4, 0, 2 \rangle$, and $\vec{b} = \langle 2, -1, 0 \rangle$. (First find an exact expression and then approximate to the nearest degree.)

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

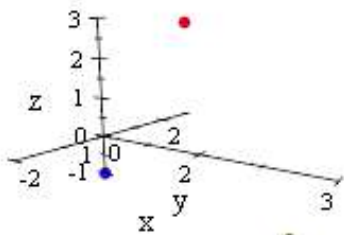
$$|\vec{a}| = \sqrt{4^2 + 0^2 + 2^2} = 2\sqrt{5}, \quad |\vec{b}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

$$\vec{a} \cdot \vec{b} = \langle 4, 0, 2 \rangle \cdot \langle 2, -1, 0 \rangle = 4 \cdot 2 + 0 \cdot (-1) + 2 \cdot 0 = 8.$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{8}{2\sqrt{5}(\sqrt{5})} = \frac{4}{5}.$$

$$\theta = \cos^{-1} \frac{4}{5} \approx 37^\circ$$

Problem #22 Find, correct to the nearest degree, the three angles of the triangle with the vertices $A(1, 0, -1)$, $B(3, -2, 0)$, $C(1, 3, 3)$.



$$\vec{AB} = \langle 3 - 1, -2 - 0, 0 + 1 \rangle = \langle 2, -2, 1 \rangle,$$

$$\vec{BC} = \langle 1 - 3, 3 + 2, 3 - 0 \rangle = \langle -2, 5, 3 \rangle,$$

$$\vec{CA} = \langle 1 - 1, 0 - 3, -1 - 3 \rangle = \langle 0, -3, -4 \rangle.$$

Recall: "If θ is the angle between \vec{a} and \vec{b} where $0 \leq \theta \leq \pi$, then $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$."

Starting with the angle θ_B between \vec{AB} and \vec{BC} :

$$\cos \theta_B = \frac{\vec{AB} \cdot \vec{BC}}{|\vec{AB}| |\vec{BC}|} = \frac{-2 \cdot 2 - 2 \cdot 5 + 1 \cdot 3}{\sqrt{2^2 + 2^2 + 1^2} \sqrt{2^2 + 5^2 + 3^2}} = \frac{-4 - 10 + 3}{\sqrt{9} \sqrt{38}} = \frac{-11}{3\sqrt{38}} \approx -0.595,$$

$$\cos \theta_C = \frac{\vec{BC} \cdot \vec{CA}}{|\vec{BC}| |\vec{CA}|} = \frac{-2 \cdot 0 - 5 \cdot 3 - 3 \cdot 4}{\sqrt{2^2 + 5^2 + 3^2} \sqrt{0^2 + 3^2 + 4^2}} = \frac{-15 - 12}{\sqrt{38} \sqrt{25}} = -\frac{27}{5\sqrt{38}} \approx -0.876,$$

$$\cos \theta_A = \frac{\vec{CA} \cdot \vec{AB}}{|\vec{CA}| |\vec{AB}|} = \frac{2 \cdot 0 + 3 \cdot 2 - 4 \cdot 1}{\sqrt{0^2 + 3^2 + 4^2} \sqrt{2^2 + 2^2 + 1^2}} = \frac{6 - 4}{\sqrt{25} \sqrt{9}} = \frac{2}{15} \approx 0.133.$$

$$\text{Therefore, } \theta_B = \cos^{-1} \left(-\frac{11}{3\sqrt{38}} \right) \approx 2.208,$$

$$\text{Or, } \cos^{-1} \left(\frac{11}{3\sqrt{38}} \right) \approx 0.9338,$$

$$\theta_C = \cos^{-1}\left(-\frac{27}{5\sqrt{38}}\right) \approx 2.638$$

$$\text{Or, } \cos^{-1}\left(\frac{27}{5\sqrt{38}}\right) \approx 0.5033,$$

$$\theta_A = \cos^{-1}\left(\frac{2}{15}\right) \approx 1.437$$

$$\text{Or, } \cos^{-1}\left(-\frac{2}{15}\right) \approx 1.705.$$

We need $\theta_A + \theta_B + \theta_C \approx \pi \approx 3.14$.

Therefore, $\theta_B \approx 0.9338$, $\theta_C \approx 0.5033$, $\theta_A \approx 1.705$.

Problem #24

a) Determine whether the following vectors are orthogonal, parallel, or neither: $\vec{u} = \langle -3, 9, 6 \rangle$, $\vec{v} = \langle 4, -12, -8 \rangle$.

$$\vec{u} \cdot \vec{v} = -3 \cdot 4 + 9 \cdot (-12) + 6 \cdot (-8) = -168$$

$\neq 0$. Not orthogonal.

$$|\vec{u}| = \sqrt{3^2 + 9^2 + 6^2} = 3\sqrt{14}, \quad |\vec{v}| = \sqrt{4^2 + 12^2 + 8^2} = 4\sqrt{14}.$$

$$\text{Therefore, } \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-168}{3\sqrt{14} \cdot 4\sqrt{14}} = -1.$$

So, \vec{u} and \vec{v} are parallel.

b) Determine whether the following vectors are orthogonal, parallel, or neither: $\vec{u} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{v} = 2\vec{i} - \vec{j} + \vec{k}$.

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 - 1 \cdot (-1) + 2 \cdot 1 = 5$$

$\neq 0$. Not orthogonal.

$$|\vec{u}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}, \quad |\vec{v}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

$$\text{Therefore, } \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{5}{6}$$

$\neq \pm 1$. So, \vec{u} and \vec{v} are NOT parallel.

c) Determine whether the following vectors are orthogonal, parallel, or neither: $\vec{u} = \langle a, b, c \rangle$, $\vec{v} = \langle -b, a, 0 \rangle$.

$$\vec{u} \cdot \vec{v} = a \cdot (-b) + b \cdot a + c \cdot 0 = 0.$$

Orthogonal, so not parallel.

Problem #30 Find the acute angle between the lines $x + 2y = 7$ and $5x - y = 2$.

We need points on the lines to generate vectors for us to manipulate.

For the first line, it intersects the axes at points $(7, 0)$ and $(0, \frac{7}{2})$. For the second line we have points $(\frac{2}{5}, 0)$ and $(0, -2)$.

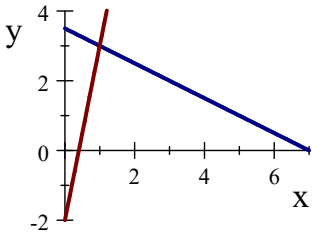
Therefore, we can define vectors $\vec{u} = 2 \cdot \langle 7 - 0, 0 - \frac{7}{2} \rangle = \langle 14, -7 \rangle$ and $\vec{v} = 5 \cdot \langle \frac{2}{5} - 0, 0 + 2 \rangle = \langle 2, 10 \rangle$, for the two lines respectively.

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{14 \cdot 2 + (-7) \cdot 10}{\sqrt{14^2 + 7^2} \sqrt{2^2 + 10^2}} = \frac{28 - 70}{\sqrt{245} \sqrt{104}} = \frac{-42}{7\sqrt{5} \cdot 2\sqrt{26}} = \frac{-3}{\sqrt{130}}.$$

$$\text{Therefore, } \theta_1 = \cos^{-1}\left(\frac{-3}{\sqrt{130}}\right) \approx 1.8375$$

$\frac{\pi}{2} \approx 1.57$, so θ_1 is the obtuse angle.

So the acute angle must be $\theta_2 = \pi - \cos^{-1}\left(\frac{-3}{\sqrt{130}}\right) \approx 1.3045$.



$$x + 2y = 7 \text{ and } 5x - y = 2$$

Problem #42 Given $\vec{a} = \langle -2, 3, -6 \rangle$ and $\vec{b} = \langle 5, -1, 4 \rangle$, find the scalar and vector projections of \vec{b} onto \vec{a} .

Scalar Projection of \vec{b} onto \vec{a} :

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-2 \cdot 5 + 3 \cdot (-1) - 6 \cdot 4}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{-10 - 3 - 24}{\sqrt{49}} = \frac{-37}{7}.$$

Vector Projection of \vec{a} onto \vec{b} :

$$\text{proj}_a \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} = \frac{-37}{7} \frac{\langle -2, 3, -6 \rangle}{\sqrt{2^2 + 3^2 + 6^2}} = -\frac{37}{49} \langle -2, 3, -6 \rangle.$$

Problem #50 A tow truck drags a stalled car along the road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N . How much work is done by the truck in pulling the car 1 km ?

$$W = \vec{F} \cdot \vec{D} = |\vec{F}| |\vec{D}| \cos \theta.$$

Observe that the units for Newtons are $\frac{\text{kg} \cdot \text{m}}{\text{s}^2}$, and the units for work (Joules) are $\frac{\text{kg} \cdot \text{m}^2}{\text{s}^2}$.

Therefore, to maintain consistent units, we convert the distance from kilometers to meters.

$$1 \text{ km} = 1000 \text{ m}$$

$$W = |\vec{F}| |\vec{D}| \cos(30^\circ) = 1500 \cdot 1000 \cdot \cos \frac{\pi}{6} \approx 1,299,038 \text{ J}.$$
