MATH 1272: Calculus II

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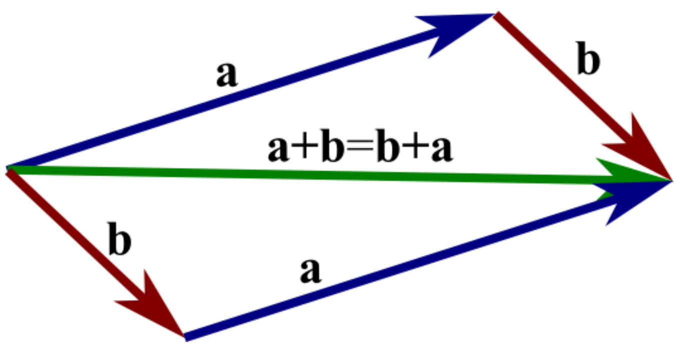
12.2 - Vectors

Review:

Vector: a mathematical object that consists of a **direction** and a **magnitude**. A vector, notated $\vec{v}$ or $\overrightarrow{AB}$, is often drawn as an arrow starting at an **initial point** A (the tail), and ending at a **terminal point** B (the tip). The magnitude is then the length of the arrow, and the direction of the vector is the direction in which the arrow is pointing. However, two vectors are considered equivalent if they share the same magnitude and direction, but have a different initial point A (and therefore also have a different B). We can also denote $\vec{v}$ with a bolded letter $\mathbf{v}$.

Zero Vector: denoted as $\mathbf{0}$ or $\vec{0}$, has length 0. It is the only vector with no specific direction.



Vector Addition (**Triangle Law**): If $\vec{a}$ and $\vec{b}$ are vectors positioned so that the initial point of $\vec{b}$ is at the terminal point of $\vec{a}$ (see image above), then the sum $\vec{a} + \vec{b}$ is the vector from the initial point of $\vec{a}$ to the terminal point of $\vec{b}$.

Scalar: A scalar is a quantity (unlike a vector) that has magnitude, but no direction. The scalars we use most frequently in this course are real numbers.

Scalar Multiplication: If c is a scalar and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is a vector whose length is $|c|$ times the length of $\vec{v}$, and whose direction is the same as $\vec{v}$ if $c > 0$, and is opposite to $\vec{v}$ if $c < 0$. If $c = 0$ or $\vec{v} = \vec{0}$, then $c\vec{v} = \vec{0}$.

Parallel Vectors: Note that two nonzero vectors are parallel if they are scalar multiples of one another.

Negative of $\vec{v}$: The negative of a vector: $(-1)\vec{v} = -\vec{v}$ has the same length as the original vector $\vec{v}$, but it points in the opposite direction. Observe that $\vec{u} - \vec{v} = \vec{u} + \left(-\vec{v}\right)$.

Components

In two dimensions (and similarly in three dimensions), if the initial point of a vector $\vec{a}$ is at the origin, then the terminal point has coordinates of the form (a_1, a_2). These coordinates are called the **components of** $\vec{a}$, and we can write the vector as $\vec{a} = \langle a_1, a_2 \rangle$. Observe the notational difference between a point $(\bullet, \bullet)$ and a vector $\langle \bullet, \bullet \rangle$.

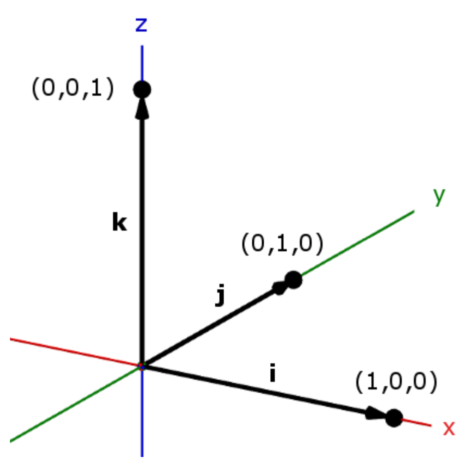
Recall that a vector is only defined by its magnitude and direction, not by its position! So when we represent a vector at a particular position, we call this a **representation** of a vector, and particularly, when positioned such that the initial point is at the origin, we call this the **Position Vector Representation**.
Vector Representation: Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\overrightarrow{AB}$ is $\vec{a} = \left\langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \right\rangle$.

Magnitude/Length of $\vec{a}$: Written as $\left|\vec{a}\right|$, we use the Pythagorean theorem to say that $\left|\vec{a}\right| = \sqrt{a_1^2 + a_2^2}$ for two dimensions, and similarly for three (or more) dimensions.

Algebraic Manipulations

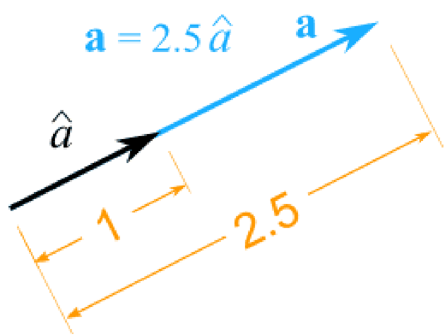
If $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$, and $\vec{d} = \langle d_1, d_2 \rangle$, and if c and e are scalars, then

- $\vec{a} + \vec{b} = \left\langle a_1 + b_1,\ a_2 + b_2 \right\rangle$, ♦ $\vec{a} - \vec{b} = \left\langle a_1 - b_1,\ a_2 - b_2 \right\rangle$,
- $c\vec{a} = \left\langle ca_1,\ ca_2 \right\rangle$; and similarly for three (or more) dimensions.
- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ Commutativity,
- $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$ Vector Associativity,
- $\vec{a} + \vec{0} = \vec{a}$ Additive Identity, ♦ $\vec{a} + \left(-\vec{a}\right) = \vec{0}$ Additive Inverse,
- $c\left(\vec{a} + \vec{b}\right) = c\vec{a} + c\vec{b}$ Scalar Distributivity,
- $(c + e)\vec{a} = c\vec{a} + e\vec{a}$ Vector Distributivity,
- $(ce)\vec{a} = c\left(e\vec{a}\right)$ Scalar Associativity ♦ $1\vec{a} = \vec{a}$ Multiplicative Identity.



Standard Basis Vectors: Let $\vec{i} := \langle 1, 0, 0 \rangle$, $\vec{j} := \langle 0, 1, 0 \rangle$, $\vec{k} := \langle 0, 0, 1 \rangle$. Observe that $\left|\vec{i}\right| = \left|\vec{j}\right| = \left|\vec{k}\right| = 1$, and any vectors in $\mathbb{R}^3$ can be expressed in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$. For instance, $\langle 1, -2, 6 \rangle = \vec{i} - 2\vec{j} + 6\vec{k}$. Observe that a similar situation is true for $\mathbb{R}^2$ where $\vec{i} := \langle 1, 0 \rangle$ and $\vec{j} := \langle 0, 1 \rangle$.

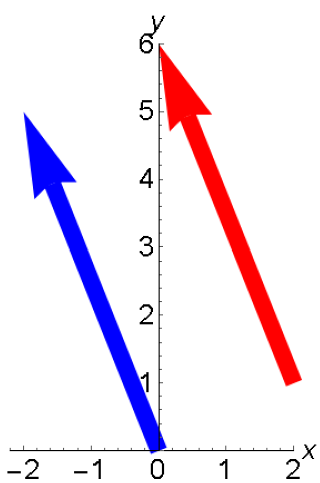
Unit Vector: A unit vector is a vector whose length is 1. This is the case for all of the standard basis vectors. If you have a vector $\vec{a}$ which is not of unit length ($|\vec{a}| \neq 1$), you can make it a unit vector without changing the direction in which the vector is pointing. You do this by dividing out the magnitude: $\frac{\vec{a}}{|\vec{a}|}$ is a **unit vector in the direction of** $\vec{a}$.



A force is often represented by a vector $\vec{f}$ because force has both magnitude and the direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces ($\vec{f}_1 + \vec{f}_2 + \vec{f}_3 = \vec{f}_{\textbf{resultant}}$).

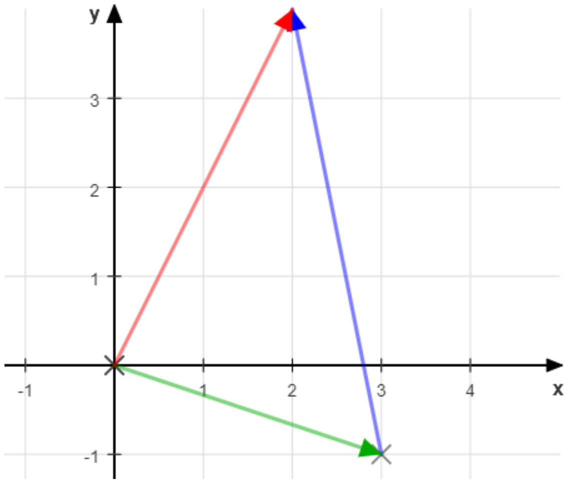
Problem #12 Let $A(2,1)$ and $B(0,6)$ be points in $\mathbb{R}^2$. Find a vector $\vec{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

$\overrightarrow{AB} = \langle 0-2,\ 6-1 \rangle = \langle -2,\ 5 \rangle$.



Problem #16 Find the sum of the vectors $\langle 3,-1 \rangle$ and $\langle -1,5 \rangle$, and illustrate this geometrically.

$\langle 3,-1 \rangle + \langle -1,5 \rangle = \langle 3+(-1),\ (-1)+5 \rangle = \langle 2,\ 4 \rangle$.



Problem #22 Let $\vec{a} = 2\vec{i} - 4\vec{j} + 4\vec{k}$, and $\vec{b} = 2\vec{j} - \vec{k}$.
Find $\vec{a} + \vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$, and $|\vec{a} - \vec{b}|$.

$$\vec{a} + \vec{b} = (2+0)\vec{i} + (-4+2)\vec{j} + (4-1)\vec{k} = 2\vec{i} - 2\vec{j} + 3\vec{k} = (2,-2,3).$$

$$2\vec{a} + 3\vec{b} = 2(2,-4,4) + 3(0,2,-1) = (4,-8,8) + (0,6,-3) = (4,-2,5).$$

$$|\vec{a}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = 6$$

$$|\vec{a} - \vec{b}| = \left|\left(2\vec{i} - 4\vec{j} + 4\vec{k}\right) - \left(2\vec{j} - \vec{k}\right)\right| = |(2,-4,4) - (0,2,-1)| = |(2,-6,5)|$$

$$= \sqrt{2^2 + (-6)^2 + 5^2} = \sqrt{4+36+25} = \sqrt{65}.$$

Problem #24 Find a unit vector that has the same direction as $\langle -4,2,4 \rangle$.

$$|\langle -4,2,4 \rangle| = \sqrt{(-4)^2 + 2^2 + 4^2} = 6$$

Unit vector: $\frac{1}{6}\langle -4,2,4 \rangle = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$.

Problem #30 If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

Converting from degrees to radians, $(38) \cdot \frac{\pi}{180} = \frac{19}{90}\pi$

Unit vector (gives us the direction of the pull): $(x, y) = (\cos(\frac{19}{90}\pi),\ \sin(\frac{19}{90}\pi))$.

On unit circle we have: $|(x, y)| = \sqrt{x^2 + y^2} = 1$. But since we need the magnitude of the force to be 50 instead of 1 …

Adding in magnitude, we have:
Force = $50(x, y) = (50 \cdot \cos(\frac{19}{90}\pi), 50 \cdot \sin(\frac{19}{90}\pi)) \approx (39.4,\ 30.8) = (\mathit{Horizontal}, \mathit{Vertical})$.

FYI: Newton $= 1\ kg \cdot m/s^2$.