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11.7 - Strategy for Testing Series

Review:

Strategies for Testing Series Convergence:

♦ If the series is of the form $\Sigma \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

♦ If the series has the form Σar^{n-1} or Σar^n, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

♦ If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function (e.g., $\frac{n}{n^3+4}$) or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-**series**. The **Comparison Tests** apply only to series with positive terms, but if Σa_n has some negative terms, then we can apply the Comparison Test to $\Sigma |a_n|$ and test for absolute convergence.

♦ If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the **Test for Divergence** should be used.

♦ If the series is of the form $\Sigma(-1)^n b_n$, then the **Alternating Series Test** is an obvious possibility.

♦ Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the **Ratio Test**. Bear in mind that $|\frac{a_{n+1}}{a_n}| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

♦ If the series is of the form $(b_n)^n$, then the **Root Test** may be useful.

♦ If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the **Integral Test** is effective (assuming the hypotheses of this test are satisfied).

Problem #2 Test the series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ for convergence and divergence.

Of the form $(b_n)^n$.

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{2n+1}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{2}{n} + \frac{1}{n^2}\right) = 0$$

$$< 1.$$

So the series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ converges by the root test.

Problem #18 Test the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ for convergence and divergence.

$b_n := \frac{1}{\sqrt{n}-1}$, for $n \geq 2$.

$\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} b_n = 0$.

So $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the alternating series test.

Problem #38 Test the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ for convergence and divergence.

Use the limit comparison test with $a_n = \sqrt[n]{2} - 1$ and $b_n = \frac{1}{n}$.

Then, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{2^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim\limits_{x\to\infty} \frac{2^{\frac{1}{x}}-1}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{2^{\frac{1}{x}} \cdot \ln 2 \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \left(2^{\frac{1}{x}} \cdot \ln 2\right)$

$= 1 \cdot \ln 2 > 0.$

So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternatively, observe that: $(\sqrt{a} - 1)(\sqrt{a} + 1) = a - 1$.
And: $(\sqrt[3]{a} - 1)(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1) = a - 1$.
And more generally: $(\sqrt[n]{a} - 1)(a^{\frac{n-1}{n}} + \ldots + a^{\frac{2}{n}} + a^{\frac{1}{n}} + 1) = a - 1$.

So: $\sqrt[n]{2} - 1 = \frac{1}{2^{\frac{n-1}{n}} + 2^{\frac{n-2}{n}} + \ldots + 2^{\frac{1}{n}} + 1}$

$\geq \frac{1}{2+2+\ldots+2+1} > \frac{1}{2n}$.

And since $\Sigma \frac{1}{2n} = \frac{1}{2} \Sigma \frac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ by the comparison test.