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11.4 - Comparison Tests

Review:

Comparison Test: Suppose that Σa_n and Σb_n are series with positive terms.
- If Σb_n is convergent and $a_n \le b_n$ for all n, then Σa_n is also convergent.
- If Σb_n is divergent and $a_n \ge b_n$ for all n, then Σa_n is also divergent.

Common Σb_n series used with the comparison test:
- p-series ($\Sigma \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise).
- geometric series (Σar^{n-1} converges if $|r| < 1$, and diverges otherwise).

Even though the requirement in the comparison test is that $a_n \le b_n$ or $a_n \ge b_n$ for all n, we can relax this requirement a bit. Observe that the convergence of a series is not affected by a finite number of terms. Therefore, at the beginning of the series, it is allowed that there be a finite number of terms not satisfying the inequality. Specifically, we need only verify that the inequality holds for $n \ge N$, where N is some fixed integer (i.e., eventually).

Limit Comparison Test: Suppose that Σa_n and Σb_n are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

Estimating Sums: Having just shown that Σa_n is less than the convergent series Σb_n, we now have a convenient way to estimate Σa_n. Let $R_n := s - s_n = a_{n+1} + a_{n+2} + \ldots$ be the remainder for Σa_n and $T_n = t - t_n$ be the remainder for Σb_n. Since $a_n \le b_n$ for all n, we must then have $R_n \le T_n$. Using the methods for estimating T_n we learned before, we can therefore estimate R_n, and therefore s.

Problem #2 Suppose Σa_n and Σb_n series with positive terms and Σb_n is known to be divergent.

a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?

If $a_n > b_n$ for all n, then Σa_n is divergent. [This is part (ii) of the comparison test]

b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

We cannot say anything about Σa_n.
If $a_n < b_n$ for all n and Σb_n is divergent, then Σa_n could be convergent or divergent.

Problem #28 Determine whether the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ converges or diverges.

Observe that $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n}$ for all $n \ge 1$,

so $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Problem #39 Prove that if $a_n \ge 0$ and Σa_n converges, then Σa_n^2 also converges.

Since Σa_n converges, $\lim_{n\to\infty} a_n = 0$, so there exists N such that $|a_n| < 1$ for all $n > N$.

But since $a_n \ge 0$, then $0 \le a_n < 1$, for all $n > N$.

And: $0 \le a_n^2 \le a_n$.

Then since Σa_n converges, so does Σa_n^2 by the comparison test.