MATH 1272: **Calculus II**

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11.1 - Sequences

Review:

Sequence: A list of numbers written in a definite order: $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$
A sequence can be thought of as a function whose domain is the set of positive integers. In other words, $f(1) = a_1,\ f(2) = a_2,$ and so on.
The sequence $\left\{a_1,\ a_2,\ a_3,\ a_n,\ \ldots\right\}$ is also sometimes written as $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$.
For example: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ or $a_n = \frac{n}{n+1}$ or $\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4}, \ldots,\ \frac{n}{n+1}, \ldots\right\}$.

Limit of a sequence $\{a_n\}$: If we can make the terms a_n as close to some finite number L as we would like by taking n sufficiently large (for example for the sequence $a_1 = 1,\ a_2 = 0.1,\ a_3 = 0.01,$ it looks like we can make a_n as close to $L = 0$ as we'd like), then we say the sequence $\{a_n\}$ has the **limit** L and we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$. If $\lim\limits_{n\to\infty} a_n$ exists, we say that the sequence **converges**. Otherwise, we say the sequence **diverges**.

Precise Definition of the Limit of a Sequence: A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding integer N such that for all $n > N$, we have $|a_n - L| < \varepsilon$.

Integer/Real Domain Limit Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and we further define the sequence $\{a_1, a_2, \ldots\}$ where $a_n = f(n)$ (we are plugging the index in for the value of x), then $\lim\limits_{n\to\infty} a_n = L$.

Infinite Limit Definition: $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positive number M, there is an integer N such that for all $n > N$, we have $a_n > M$.

Limit laws for Sequences:

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n,$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n, \qquad \lim\limits_{n\to\infty} c = c,$
- $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n,$
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n},$ as long as $\lim\limits_{n\to\infty} b_n \neq 0,$
- $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ as long as $p > 0$ and $a_n > 0$.

Squeeze Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Zero Absolute Limit Theorem: If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Sequential Limit, **Continuous Function Theorem**: If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

Convergence of r^n: $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. In fact:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 \text{ if } -1 < r < 1, \\ 1 \text{ if } r = 1. \end{cases}$$

Definitions of Increasing/Decreasing/Monotonic Sequences: $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < \ldots$ The sequence is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotone** if it is either increasing or decreasing.

Sequential Boundedness: A sequence $\{a_n\}$ is **bounded above** if there is a number of M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number of m such that $m \leq a_n$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Completeness Axiom for $\mathbb{R}$ (the real numbers): If S is a nonempty set of real numbers that has an **Upper Bound** M ($x \leq M$ for all x in S), then S has a **Least Upper Bound** b (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leq M$.).

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

Problem #3 List the first five terms of the sequence $a_n = \frac{2n}{n^2+1}$.

$$\left\{\frac{2}{1+1}, \frac{4}{4+1}, \frac{6}{9+1}, \frac{8}{16+1}, \dots\right\} = \left\{1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \dots\right\}$$

Problem #10 List the first five terms of the sequence $a_1 = 6, \quad a_{n+1} = \frac{a_n}{n}$.

$a_2 = \frac{a_1}{1} = \frac{6}{1} = 6.$

$a_3 = \frac{a_2}{2} = \frac{6}{2} = 3.$

$a_4 = \frac{a_3}{3} = \frac{3}{3} = 1. \qquad a_5 = \frac{a_4}{4} = \frac{1}{4}. \qquad \left\{6, 6, 3, 1, \frac{1}{4}, \dots\right\}.$

Problem #16 Find a formula for the general term a_n of the sequence $\{5, 8, 11, 14, 17, \dots\}$, assuming that the pattern of the first few terms continues.

Each term is larger than the previous term by 3, so:

$a_n = a_1 + d(n-1)$

$= 5 + 3(n-1) = 3n + 2.$

Problem #24 Determine whether the sequence $a_n = \frac{n^3}{n^3+1}$ converges or diverges. If it converges, find the limit.

Simplifying by dividing the numerator and the denominator by n^3:

$\dfrac{\frac{n^3}{n^3}}{\frac{n^3+1}{n^3}} = \dfrac{1}{1+\frac{1}{n^3}}$, so $a_n \to \frac{1}{1+0} = 1$ as $n \to \infty$. Converges

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Problem #48 Determine whether the sequence $a_n = \frac{\sin 2n}{1+\sqrt{n}}$ converges or diverges. If it converges, find the limit.

$|a_n| \leq \frac{1}{1+\sqrt{n}}$

And $\lim\limits_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$

So $\frac{-1}{1+\sqrt{n}} \leq a_n \leq \frac{1}{1+\sqrt{n}}$

$\Rightarrow \quad \lim\limits_{n\to\infty} a_n = 0$ by the squeeze theorem. So the sequence converges.

Problem #71 Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$.

Because all of its terms lie between 5 and 8, then $\{a_n\}$ is a bounded sequence.

By the monotone sequence theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L.

L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.