MATH 1272: Calculus II

**Discussion Instructor**: Jodin Morey moreyjc@umn.edu
**Discussion Session Website**: math.umn.edu/~moreyjc

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# 9.4 - Models for Population Growth

## Review:

**Natural Growth and Decay** (decay of radioactive elements, growth of bacteria, etc.): $\frac{dP}{dt} = kP$ where $k$ is a constant.

Using separation of variables: $\int \frac{dP}{P} = \int k dt \quad \Rightarrow \quad \ln|P| = kt + c$

$\Rightarrow \quad |P| = e^{kt+c} = e^c e^{kt} \quad \Rightarrow \quad P = P_0 e^{kt}$ where $P_0 = \pm e^c$ or $P_0 = 0$.

**Logistic Equation**: $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ where $P$ is the population, $M$ is the limiting population, and $k$ is a constant.

**Solution to the logistic equation** (accomplished using separation of variables):

$P(t) = \frac{M}{1+Ae^{-kt}}$, where $A = \frac{M-P_0}{P_0}$. [Equation 7]

**Harvesting a Logistic Population**: $\frac{dP}{dt} = kP(1 - \frac{P}{M}) - c$.

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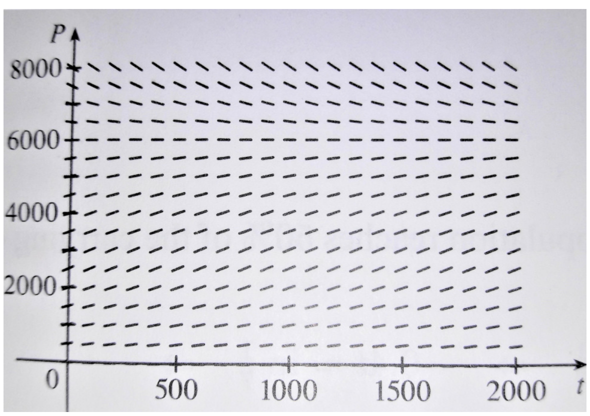
**Problem #2** Suppose that a population grows according to a logistic model with carrying capacity $6,000$ and $k = 0.0015$ per year.

**a) Write the logistic differential equation for these data.**

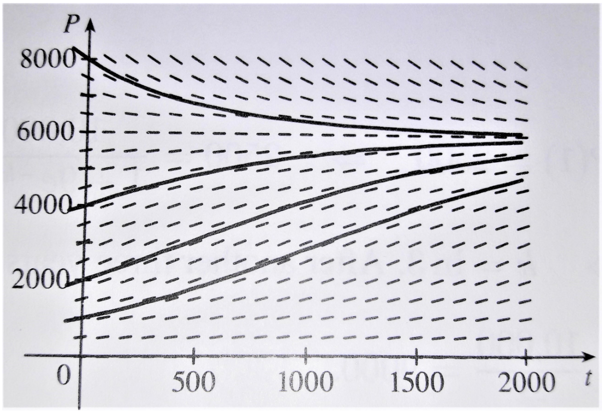
$M = 6000$ and $k = 0.0015$ implies $\frac{dP}{dt} = (0.0015)P\left(1 - \frac{P}{6000}\right)$.

**b) Draw a direction field. What does it tell you about the solution curves?**




All of the solution curves approach $6,000$ as $t \to \infty$

**c) Use the direction field to sketch the solution curves for initial populations of 1,000, 2,000, 4,000, and 8,000. What can you say about the concavity of these curves? What is the significance of the inflection points?**



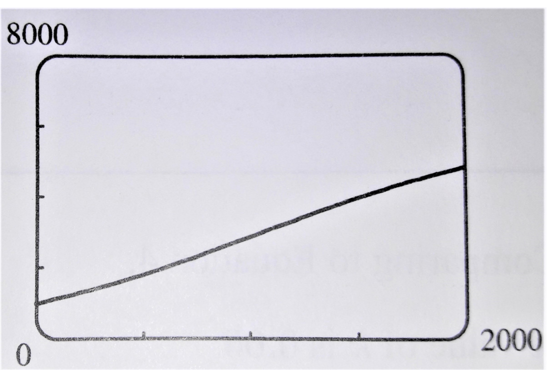

The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

**e) If the initial population is 1000, write a formula for the population after $t$ years. Use it to find the population after 50 years.**

Using equation 7 from the text with $M = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P(t) = \frac{M}{1+Ae^{-kt}} = \frac{6000}{1+Ae^{-0.0015t}}$, where $A = \frac{M-P_0}{P_0} = \frac{6000-1000}{1000} = 5$.

Therefore, $P(50) = \frac{6000}{1+5e^{-0.0015(50)}} \approx 1064$.

**f) Graph the solution in part $(e)$ and compare with the solution curve you sketched in part $(c)$.**




**Problem #4** Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2$ with $P(0) = 50$ where $t$ is measured in years.

**a) What is the carrying capacity?**

$\frac{dP}{dt} = 0.4P(1 - 0.0025P)$ (since $\frac{0.001}{0.4} = 0.0025$)

$= 0.4P\left(1 - \frac{P}{400}\right)$ (since $0.0025^{-1} = 400$)

Therefore, by equation 1 from the text, we have that $k = 0.4$, and the carrying capacity is 400.

**b) What is $P'(0)$?**

Using the fact that $P(0) = 50$ people and the formula for $\frac{dP}{dt}$, we get $P'(0) = \frac{dP}{dt}\Big|_{t=0} = 0.4(50) - 0.001(50)^2 = 17.5$ people per year.

**c) When will the population reach 50% of the carrying capacity?**

From Equation 7 in the text, $A = \frac{M-P_0}{P_0} = \frac{400-50}{50} = 7$, so $P = \frac{400}{1+7e^{-0.4t}}$.

The population reaches 50% of the carrying capacity (200), when $\frac{400}{1+7e^{-0.4t}} = 200$.

Solving for $t$: $1 + 7e^{-0.4t} = 2 \quad \Rightarrow \quad e^{-0.4t} = \frac{1}{7} \quad \Rightarrow \quad -0.4t = \ln\frac{1}{7}$

$\Rightarrow \quad t = \frac{\ln\frac{1}{7}}{-0.4} \approx 4.86$ years.

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**Problem** #10 Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be $10,000$. The number of fish tripled in the first year.
a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after $t$ years.

Interpreting the information given: $P(0) = P_0 = 400,\ P(1) = 1200,$ and $M = 10,000.$

From the solution to the logistic differential equation (Equation 7) $P(t) = \frac{P_0 M}{P_0+(M-P_0)e^{-kt}}$, we get $P(t) = \frac{400(10{,}000)}{400+(9600)e^{-kt}} = \frac{10{,}000}{1+24e^{-kt}}$.

$P(1) = 1200 \Rightarrow \quad 1 + 24e^{-k} = \frac{100}{12} \quad \Rightarrow \quad e^{k} = \frac{288}{88} \quad \Rightarrow \quad k = \ln\frac{36}{11.}$

So, $P(t) = \frac{10{,}000}{1+24e^{-t\ln\frac{36}{11}}} = \frac{10{,}000}{1+24\cdot\left(\frac{11}{36}\right)^t}$.

**b) How long will it take for the population to increase to 5,000**?

$5000 = \frac{10{,}000}{1+24\left(\frac{11}{36}\right)^t} \quad \Rightarrow \quad 24\left(\frac{11}{36}\right)^t = 1 \quad \Rightarrow \quad t\ln\frac{11}{36} = \ln\frac{1}{24} \quad \Rightarrow \quad t \approx 2.68$ years.