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8.3 - Applications to Physics and Engineering

Review:

Hydrostatic Pressure and Force:
Definition: Hydrostatics is the branch of fluid mechanics that studies fluids at rest. It encompasses the study of the conditions under which fluids are at rest in stable equilibrium as opposed to fluid dynamics, the study of fluids in motion.
Let F be the hydrostatic force (the force felt on a surface within a fluid due to the pressure of that fluid at rest), P be the hydrostatic pressure (the force at a point on the surface), g be the force of gravity, ρ be the density of the fluid, A be the surface area of a plate, d be the depth of the plate below the surface, V be the volume of the fluid above the plate, and $m = \rho V = \rho dA$ be the mass of the fluid above the plate. Then, the force on the plate can be written as…

- $F = mg = \rho g d A$, where the hydrostatic pressure is:
- $P = \frac{F}{A} = \rho g d$.

For water, $\rho = 1000\,kg/m^3$ or $0.0624\,lb/ft^3$.
The hydrostatic force on a dam where w is the width of the dam, the water is h meters high, and where height is measured on a vertical x-axis is:

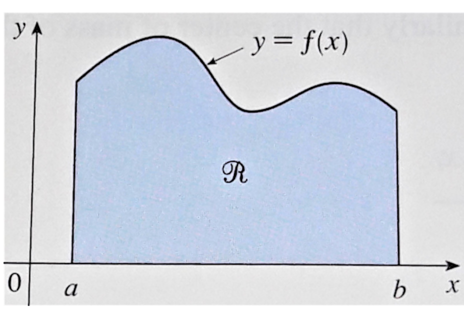
$$F = \int_0^h \rho g dA = \int_0^h 1000(9.8)x(wdx).$$

Moments and Centers of Mass:
Moments of an n-particle system about the x-axis and y-axis.

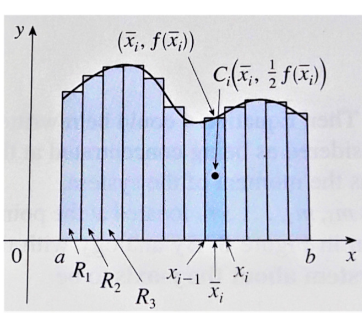
$M_x = \sum_{i=1}^n m_i y_i$ and $M_y = \sum_{i=1}^n m_i x_i$, for n particles with m_i mass.

The **center of mass** is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{m}$ $\bar{y} = \frac{M_x}{m}$
and $m = \sum m_i$, (the total mass).

For a flat plate with uniform density ρ, the center of mass is called the **centroid**.
The **symmetry principle** says that if a flat plate is symmetric about a line, then the centroid is located on the line.



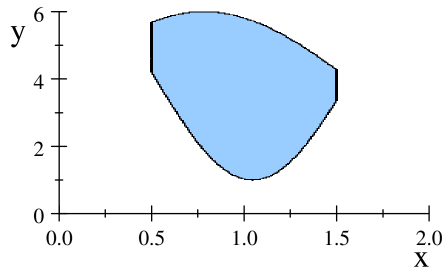
Imagine a flat plate sitting on the x-axis where the shape of the top of the plate is defined by $f(x)$ (see the figure). Next, divide it into n vertical strips where the edges of these strips are located at $\{x_0, x_1, \ldots, x_n\}$. Also, let $\bar{x_i}$ be the midpoint between x_{i-1} and x_i. For example, $\bar{x_1} = \frac{x_0+x_1}{2}$.



Then, the **moment** of the flat plate about the y-axis is: $M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \overline{x_i} f(\overline{x_i}) \Delta x = \rho \int_a^b x f(x) dx.$

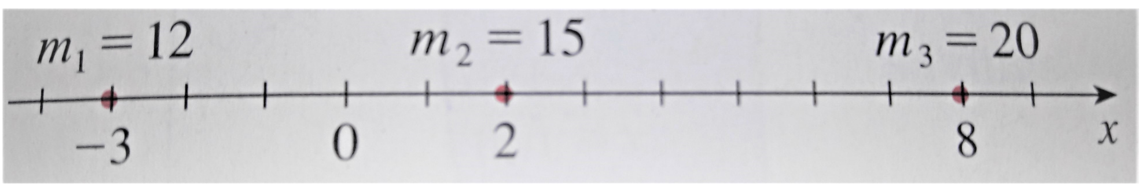
About the x-axis: $M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\overline{x_i})]^2 \Delta x = \frac{\rho}{2} \int_a^b [f(\overline{x_i})]^2 dx.$

The **center of mass** of the above plate is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{A} \int_a^b x f(x) dx \qquad \overline{y} = \frac{1}{2A} \int_a^b [f(x)]^2 dx.$



In the image above, let $f(x)$ be the upper curve, and $g(x)$ be the lower curve of the plate.
The **center of mass** is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)] dx \qquad \overline{y} = \frac{1}{2A} \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx.$

Theorem of Pappus: If you have a flat plate that is rotated about a line that is to one side of the plate, then the volume of the solid-of-revolution is $A \cdot d$ where A is the area of the plate, and d is the distance traveled by the centroid of the plate.



Problem #22 Point masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\overline{x}$.

The moment M is $m_1x_1 + m_2x_2 + m_3x_3 = 12(-3) + 15(2) + 20(8) = 154.$

The mass m is $m_1 + m_2 + m_3 = 12 + 15 + 20 = 47.$

The center of mass is $\overline{x} = \frac{M}{m} = \frac{154}{47} \approx 3.2766.$

Problem #23 The masses $m_1 = 4,\ m_2 = 2,\ m_3 = 4$ are located at the points $P_1(2,-3),\ P_2(-3,1),\ P_3(3,5)$. Find the moments M_x and M_y and the center of mass of the system.

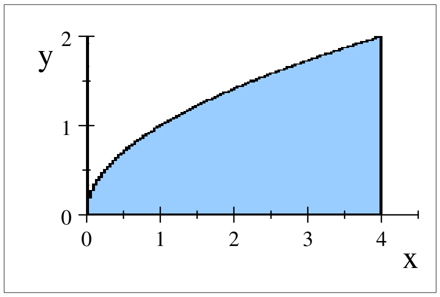
The mass is $m = \sum_{i=1}^{3} m_i = 4 + 2 + 4 = 10.$

The moment about the x-axis is $M_x = \sum_{i=1}^{3} m_i y_i = 4(-3) + 2(1) + 4(5) = 10.$

The moment about the y-axis is $M_y = \sum_{i=1}^{3} m_i x_i = 4(2) + 2(-3) + 4(3) = 14.$

The center of mass is $(\overline{x},\ \overline{y}) = \left(\frac{M_y}{m},\ \frac{M_x}{m}\right) = \left(\frac{14}{10}, \frac{10}{10}\right) = (1.4,\ 1).$

Problem #26 Sketch the region bounded by the curves $y = \sqrt{x},\ y = 0,\ x = 4,$ and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.



we will

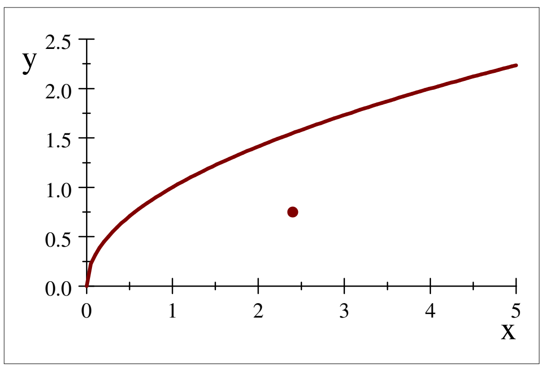
The region and the figure is “right-heavy" and “bottom-heavy," so we know that $\bar{x} > 2$ and $\bar{y} < 1$, and we might guess that $\bar{x} \approx 2.3$ and $\bar{y} \approx 0.8$.

$$A = \int_0^4 \sqrt{x}\, dx = \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \tfrac{16}{3}.$$

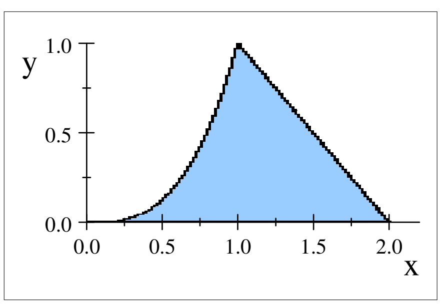
$$\bar{x} = \tfrac{1}{A}\int_0^4 x(\sqrt{x})dx = \tfrac{3}{16}\int_0^4 x^{\frac{3}{2}}dx = \tfrac{3}{16}\left[\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \tfrac{3}{40}(32-0) = \tfrac{12}{5}.$$

$$\bar{y} = \tfrac{1}{A}\int_0^4 \tfrac{1}{2}(\sqrt{x})^2 dx = \tfrac{3}{16}\int_0^4 \tfrac{1}{2}x dx = \tfrac{3}{32}\left[\tfrac{1}{2}x^2\right]_0^4 = \tfrac{3}{64}(16-0) = \tfrac{3}{4}.$$

Thus, the centroid is $(\bar{x},\bar{y}) = \left(2.4,\ 0.75\right)$.



Problem #32 Find the centroid of the region bounded by the curves $y = x^3,\ x+y = 2,\ y = 0.$



First we will want to find the area by integrating under the curve. However,our curve is determined by two functions. Where does one function end, and the other function start?

$x^3 = 2-x \Rightarrow x^3+x-2 = 0$, we could attempt to solve this cubic, or we can observe from the graph that the two functions appear to meet at $x = 1$, and indeed this is a solution to the cubic.

$$A = \int_0^1 x^3 dx + \int_1^2 (2-x)dx = \left[\tfrac{1}{4}x^4\right]_0^1 + \left[2x - \tfrac{1}{2}x^2\right]_1^2$$

$$= \tfrac{1}{4} + (4-2) - \left(2-\tfrac{1}{2}\right) = \tfrac{3}{4}.$$

$$\bar{x} = \tfrac{1}{A}\left[\int_0^1 x(x^3)dx + \int_1^2 x(2-x)dx\right] = \tfrac{4}{3}\left[\int_0^1 x^4 dx + \int_1^2 (2x-x^2)dx\right]$$

$$= \tfrac{4}{3}\left\{\left[\tfrac{1}{5}x^5\right]_0^1 + \left[x^2 - \tfrac{1}{3}x^3\right]_1^2\right\} = \tfrac{4}{3}\left[\tfrac{1}{5} + \left(4-\tfrac{8}{3}\right) - \left(1-\tfrac{1}{3}\right)\right]$$

$$= \frac{4}{3}\left(\frac{13}{15}\right) = \frac{52}{45}.$$

$$\overline{y} = \frac{1}{A}\left[\int_0^1 \frac{1}{2}(x^3)^2 dx + \int_1^2 (2-x)^2 dx\right] = \frac{2}{3}\left[\int_0^1 x^6 dx + \int_1^2 (x-2)^2 dx\right]$$

$$= \frac{2}{3}\left\{\left[\frac{1}{7}x^7\right]_0^1 + \left[\frac{1}{3}(x-2)^3\right]_1^2\right\} = \frac{2}{3}\left(\frac{1}{7} - 0 + 0 + \frac{1}{3}\right) = \frac{2}{3}\left(\frac{10}{21}\right) = \frac{20}{63}.$$

Thus the centroid is $(\overline{x}, \overline{y}) = \left(\frac{52}{45}, \frac{20}{63}\right)$.