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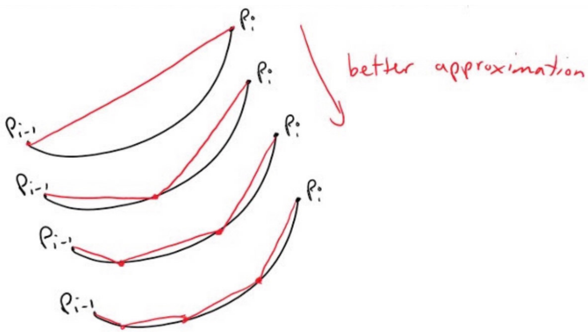
8.1 - Arc Length

Review: Length of a curve $f(x)$.



Riemann Arc Length Formula:
For $y = f(x)$ on $[a,b]$, if you choose equally spaced points $P_1, P_2, \ldots$ along the curve, then the length of the curve is…

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|, \text{ where } |P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

(the straight line distance between each pair of neighboring points).

The above is true if the limit exists (in other words, it's true if the curve is "well behaved," as it is when $f(x)$ is continuous).

Integral Arc Length Formula:
If $f'(x)$ is continuous on $[a,b]$, then the length of the curve is:

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx \quad \text{or equivalently} \quad L = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx.$$

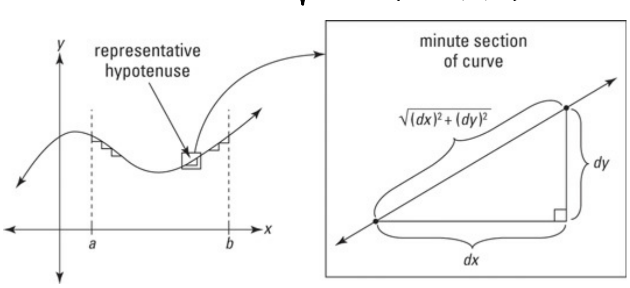
If we're integrating along the y-axis instead, the function is of the form $x = g(y)$, and the equation becomes:

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy \quad \text{or equivalently} \quad L = \int_c^d \sqrt{1 + \left(\tfrac{dx}{dy}\right)^2}\, dy.$$

The Arc Length Function: $s(z) = \int_a^z \sqrt{1 + \left(\tfrac{dx}{dt}\right)^2}\, dx.$

A mnemonic for remembering the previous equations involves the visualization in the book of the triangular relationship between ds, dx, and dy.
Particularly, since $(ds)^2 = (dx)^2 + (dy)^2$, therefore…

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2\left(1 + \frac{(dy)^2}{(dx)^2}\right)} = \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx, \text{ and } s = \int ds = \int \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx.$$



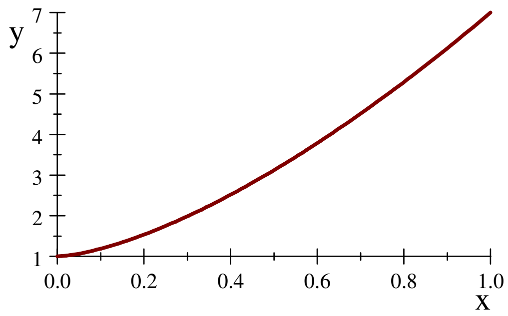
Problem #7 Find the exact length of the curve $y = 1 + 6x^{\frac{3}{2}}$, on $0 \le x \le 1$.

$$\frac{dy}{dx} = 9x^{\frac{1}{2}} \quad \Rightarrow \quad 1 + \left(\tfrac{dy}{dx}\right)^2 = 1 + 81x.$$

So $L = \int_{x=0}^{x=1} \sqrt{1+81x}\,dx$

Let $u = 1+81x,\ du = 81dx$

So $L = \int_{u=1}^{u=82} u^{\frac{1}{2}}\left(\frac{1}{81}du\right) = \frac{1}{81}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{82} = \frac{2}{243}\left(82\sqrt{82}-1\right) \approx 6.103.$



Problem #11 Find the exact length of the curve: $x = \frac{1}{3}\sqrt{y}\,(y-3)$, on $1 \le y \le 9$.

$\frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{1}{2}}$

$\frac{dx}{dy} = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}}$

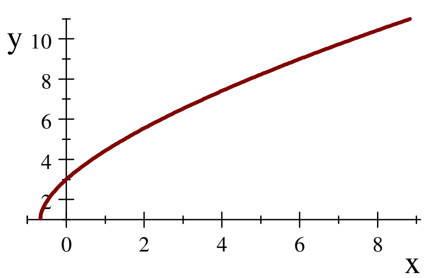
$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(\frac{1}{4}y-\frac{1}{2}+\frac{1}{4}y^{-1}\right) = \frac{1}{4}y+\frac{1}{2}+\frac{1}{4}y^{-1}$

$= \left(\frac{1}{2}y^{\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\right)^2.$!!!

So, $L = \int_1^9 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^9 \left(\frac{1}{2}y^{\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\right)dy$

$= \frac{1}{2}\left[\frac{2}{3}y^{\frac{3}{2}}+2y^{\frac{1}{2}}\right]_1^9 = \frac{1}{2}\left[\left(\frac{2}{3}\cdot 27+2\cdot 3\right)-\left(\frac{2}{3}\cdot 1+2\cdot 1\right)\right]$

$= \frac{1}{2}\left(24-\frac{8}{3}\right) = \frac{1}{2}\left(\frac{64}{3}\right) = \frac{32}{3}.$



Problem #33 Find the arc length function for the curve $y = 2x^{\frac{3}{2}}$ with starting point $P_0(1,2)$.

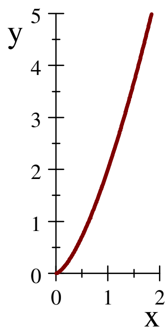
$y' = 3x^{\frac{1}{2}}$

$1+(y')^2 = 1+9x.$

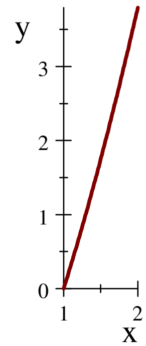
The arc length function with starting point $P_0(1,2)$ is

$$s(x) = \int_1^x \sqrt{1+9t}\,dt$$

$$= \left[\frac{2}{27}(1+9t)^{\frac{3}{2}} \right]_1^x = \frac{2}{27}\left[(1+9x)^{\frac{3}{2}} - 10\sqrt{10} \right].$$



$y = 2x^{\frac{3}{2}}$



$s(x)$