MATH 1272: **Calculus II**

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7.8 - Improper Integrals

Review:

Definition: An **improper integral** is the limit of a definite integral $\int_a^b$ as an endpoint (or both endpoints) of the interval of integration approaches either a specified real number (e.g., $b \to 7$), or positive or negative infinity (e.g., $a \to -\infty$).

Infinite Intervals [Type 1]

♦ If $\int_a^t f(x)dx$ exists (is finite) for every number $t \geq a$, then …
$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$, provided this limit exists (has a limit, and is finite).

♦ If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then …
$\int_{-\infty}^b f(x)dx = \lim_{t\to\infty} \int_t^b f(x)dx$, provided this limit exists.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

♦ If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define
$\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$

The p-test: $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.
https://www.desmos.com/calculator/ejmk1xvet2

Discontinuous Integrands [Type 2]

♦ If f is continuous on $[a,b)$ and is discontinuous at b, then …
$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$, if this limit exists (as a finite number).

♦ If f is continuous on $(a,b]$ and is discontinuous at a, then …
$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$ if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

♦ If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define $\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx.$

Comparison Test for Improper Integrals:

Assume f, g are continuous with $f(x) \geq g(x) \geq 0$ for all $x \geq a$, then:

- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Problem #2 Which of the following integrals are improper? Why?

a) $\int_0^{\frac{\pi}{4}} \tan x dx$, b) $\int_0^{\pi} \tan x dx$, c) $\int_{-1}^{1} \frac{dx}{x^2-x-2}$, d) $\int_0^{\infty} e^{-x^3} dx$.

a) Since $y = \tan x$ is defined and continuous on $[0, \frac{\pi}{4}]$, $\int_0^{\frac{\pi}{4}} \tan x dx$ is proper.

b) Since $y = \tan x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi} \tan x dx$ is a Type 2 improper integral.

c) Since $y = \frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)}$ has an infinite discontinuity at $x = -1$, then $\int_{-1}^{1} \frac{dx}{x^2-x-2}$ is a Type 2 improper integral.

d) Since $\int_0^{\infty} e^{-x^3} dx$ has an infinite interval of integrations, it is an improper integral of Type 1.

Problem #40 Determine whether $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$ is convergent or divergent. Evaluate it if it is convergent.

Integrate by parts with $u = \ln x$, $dv = \frac{dx}{\sqrt{x}}$ implies $du = \frac{dx}{x}$, $v = 2\sqrt{x}$.

$$\int_0^1 \frac{\ln x}{\sqrt{x}} dx = [2\sqrt{x}\ln x]_0^1 - 2\int_0^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{t\to 0^+} \left([2\sqrt{x}\ln x]_t^1 - 2\int_t^1 \frac{dx}{\sqrt{x}} \right)$$

$$= \lim_{t\to 0^+} \left(-2\sqrt{t}\ln t - 4[\sqrt{x}]_t^1 \right) = \lim_{t\to 0^+} (-2\sqrt{t}\ln t - 4 + 4\sqrt{t})$$

$$= -2\lim_{t\to 0^+} (\sqrt{t}\ln t) - 4.$$

Observe that $\lim_{t\to 0^+} \sqrt{t}\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-\frac{1}{2}}} \overset{L'H}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}} = \lim_{t\to 0^+} (-2\sqrt{t}) = 0.$

So $\int_0^1 \frac{\ln x}{\sqrt{x}} dx = -4$ is convergent.

Determine whether $\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}} dx$ is convergent or divergent. Evaluate it if it is convergent.

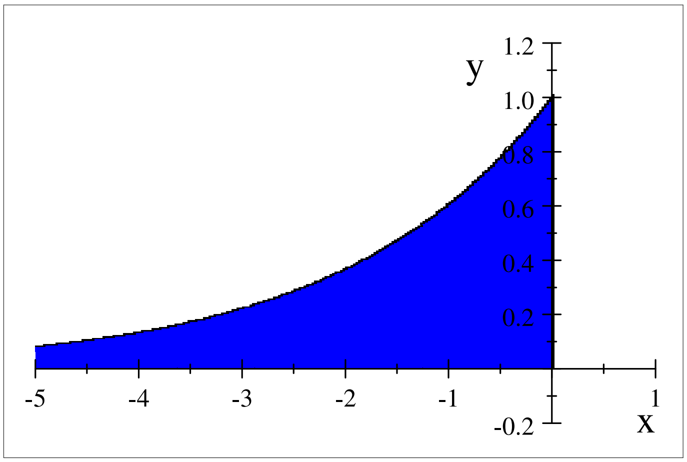
$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}} dx = \int_{-1}^{1} x^{-\frac{2}{3}} dx = \int_{-1}^{0} x^{-\frac{2}{3}} dx + \int_0^1 x^{-\frac{2}{3}} dx$$

$= \lim_{t\to 0} \int_{-1}^{t} x^{-\frac{2}{3}}\,dx + \lim_{t\to 0} \int_{t}^{1} x^{-\frac{2}{3}}\,dx = \lim_{t\to 0}\left[3x^{\frac{1}{3}}\right]_{-1}^{t} + \lim_{t\to 0}\left[3x^{\frac{1}{3}}\right]_{t}^{1}$

$= \lim_{t\to 0}\left(3t^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}\right) + \lim_{t\to 0}\left(3(1)^{\frac{1}{3}} - 3t^{\frac{1}{3}}\right)$

$= (0+3) + (3-0) = 6.$

Problem #42 Sketch the region $S = \{(x,y) : x \le 0,\ 0 \le y \le e^x\}$ and find its area (if the area is finite).



$\text{Area} = \int_{-\infty}^{0} e^x\,dx$

$= \lim_{t\to -\infty} \int_{t}^{0} e^x\,dx$

$= \lim_{t\to -\infty} [e^x]_t^0 = \lim_{t\to -\infty} (e^0 - e^t)$

$= 1 - 0 = 1.$

Problem #52 Use the comparison theorem to determine whether $\int_0^\infty \frac{\tan^{-1}x}{2+e^x}dx$ is convergent or divergent.

For $x \ge 0$, $\tan^{-1}x < \frac{\pi}{2} < 2$.

So $\frac{\tan^{-1}x}{2+e^x} < \frac{2}{2+e^x}$ ....

$< \frac{2}{e^x} = 2e^{-x}.$

Now, $I = \int_0^\infty 2e^{-x}dx = \lim_{t\to\infty} \int_0^t 2e^{-x}dx = \lim_{t\to\infty} [-2e^{-x}]_0^t = \lim_{t\to\infty} \left(-\frac{2}{e^t} + 2\right) = 2.$

So I is convergent, and by comparison, $\int_0^\infty \frac{\tan^{-1} x}{2+e^x}\,dx < I$ is also convergent.