Probability Theory: Activity 19 Solutions

Let X follow a uniform distribution on the interval (5, 13).

1. What is the mean and variance of X?

$$E(X) = \int_5^{13} x\frac{1}{8} = \frac{x^2}{16}\Big|_5^{13} = \frac{1}{16}(13^2 - 5^2) = 9$$
$$E(X^2) = \int_5^{13} x^2\frac{1}{8} = \frac{x^3}{24}\Big|_5^{13} = \frac{1}{24}(13^3 - 5^3) = 86\frac{1}{3}$$

So the mean is 9 and the variance is $Var(X) = 86\frac{1}{3} - 9^2 = \frac{16}{3}$

2. What is the probability that X is between 8 and 9.5?

$$P(8 < X < 9.5) = \int_8^{9.5} \frac{1}{8} = \frac{x}{8}\Big|_8^{9.5} = 0.1875$$

3. We observe 36 independent random variables which all follow a uniform distribution on the interval (5,13). What distribution does the sample mean follow? What is its mean and variance?

The CLT tells us that this follows a Normal Distribution with mean 9 and variance $\frac{\frac{16}{3}}{36} = \frac{4}{27}$.

4. What is the probability that the sample mean is between 8 and 9.5?

We know the sample mean follows a Normal. So we have:

$$\begin{aligned} P(8 < \bar{X}_n < 9.5) &= P\left(\frac{8-9}{\sqrt{4/27}} < Z < \frac{9.5-9}{\sqrt{4/27}}\right) \\ &= P(-2.59 < Z < 1.29) \\ &= \Phi(1.29) - \Phi(-2.59) \\ &= \Phi(1.29) - (1 - \Phi(2.59)) \\ &= 0.9015 - (1 - 0.9953) = 0.8968 \end{aligned}$$

5. Why are the answers to 2 and 4 different? Which is larger? Why?

In the first case, we are looking at the distribution of a single random variable X, which has a large variance of 16/3. So it has a relatively low probability of being in between 8 and 9.5. In the second case, we are looking at the distribution of of the sample mean of 36 random variables. The sample mean is much more likely to be near the mean of 9 than far from it, and its variance is much smaller. As a result, the probability of being between 8 and 9.5 is much higher for the sample mean.