Probability Theory: Activity 18 Solutions

To study the prevalence of a disease in a several cities of interest within a certain county, we pick a city at random, then pick a random sample of n people from that city. Let p be the proportion of diseased people in the chosen city, and let X be the number of diseased people in the sample. Since every city has a different disease proportion, p is a random variable. Suppose that $p \sim Unif(0,1)$. Also, assume that, given p, assume $X|p$ follows a binomial(n,p).

Find $E(X)$ and $Var(X)$.

1. Use the properties of the binomial to find $E(X|p)$ and $Var(X|p)$.

$$E(X|p) = np$$

and

$$Var(X|p) = np(1-p).$$

This is just from the properties of the binomial distribution.

2. Use the fact that $p \sim$ Unif(0,1) to find $E(p)$ and $Var(p)$.

$$E(p) = \frac{1}{2}.$$

$$E(p^2) = \int_0^1 x^2 = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

$$Var(p) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

3. Find $E(X)$ using Adams law: $E(X) = E(E(X|p))$.

$$E(X) = E(E(X|p)) = E(np) = nE(p) = \frac{n}{2}$$

4. Find $E(Var(X|p))$.

$$E(Var(X|p)) = E(np(1-p)) = nE(p-p^2) = n(E(p) - E(p^2)) = n\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{n}{6}$$

5. Find $Var(E(X|p))$.

$$Var(E(X|p)) = Var(np) = n^2 Var(p) = \frac{n^2}{12}$$

6. Find $Var(X)$ using Eve's law: $Var(X) = E(Var(X|p)) + Var(E(X|p))$.

$$Var(X) = E(Var(X|p)) + Var(E(X|p)) = \frac{n}{6} + \frac{n^2}{12}.$$