

## Probability Theory: Activity 17 Solutions

1. Let  $X$  and  $Y$  be random variables with the following joint PF:

$$f(x, y) = \begin{cases} 12x^2 & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of  $X$ .

$$f_X(x) = \int_x^1 12x^2 dy = 12x^2 y \Big|_x^1 = 12x^2 - 12x^3$$

(b) Find the conditional distribution  $f_{Y|X}(y|x)$ .

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{12x^2}{12x^2 - 12x^3} = \frac{1}{1-x}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{1-x} & x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find  $E(Y|X)$ .

$$E(Y|X) = \int y f_{Y|X}(y|x) dy = \int_x^1 \frac{y}{1-x} dy = \frac{y^2}{2(1-x)} \Big|_x^1 = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

(d) Let  $g(x) = E(Y|X)$ . Find  $E(g(x))$ .

$$E(g(x)) = \int g(x) f_X(x) dx = \int_0^1 \frac{1+x}{2} (12x^2 - 12x^3) dx = \int_0^1 6x^2 - 6x^4 dx$$

$$E(g(x)) = 2x^3 - \frac{6}{5}x^5 \Big|_0^1 = \frac{4}{5}$$

(e) The marginal distribution of  $Y$  is:

$$f_Y(y) = \begin{cases} 4y^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find  $E(Y)$ .

$$E(Y) = \int_0^1 y 4y^3 dy = \frac{4}{5} y^5 \Big|_0^1 = \frac{4}{5}$$