Probability Theory: Activity 10 Solutions

1. Let X and Y be random variables with the following joint distribution:

	Y		
X	0	2	4
1	0.1	0.05	0.05
2	0.1	0.1	0.2
3	0.2	0.05	0.15

   (a) Find $E(XY)$.

$$\begin{aligned} \mathrm{E}(XY) &= \sum_x \sum_y xyf(x,y) \\ &= 0xf(x,0) + 2xf(x,2) + 4xf(x,4) \\ &= 2(1)(0.05) + 2(2)(0.1) + 2(3)(0.05) + 4(1)(0.05) + 4(2)(0.2) + 4(3)(0.15) \\ &= 0.1 + 0.4 + 0.3 + 0.2 + 1.6 + 1.8 \\ &= 4.4 \end{aligned}$$

   (b) $E(X) = 2.2$ and $E(Y) = 2$. Find the covariance of X and Y.

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 4.4 - 2.2(2) = 4.4 - 4.4 = 0$$

   So they are uncorrelated.

2. Let X and Y be continuous random variables with the following joint PF:

$$f(x,y) = \begin{cases} 12y^2 & \text{for } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal distribution of X.

$$\int_0^x 12y^2 dy = 4y^3|_0^x = 4x^3$$

$$f_X(x) = \begin{cases} 4x^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Find the expectation of X.

$$\mathrm{E}(X) = \int_0^1 xf_X(x) = \int_0^1 x4x^3 = \frac{4}{5}x^5|_0^1 = \frac{4}{5}$$

   (c) Given that $E(Y) = \frac{3}{5}$, find the covariance of X and Y.

$$\mathrm{E}(XY) = \int_0^1 \int_0^x xy12y^2 dydx = \int_0^1 3xy^4 dx|_0^x = \int_0^1 3x^5 dx = \frac{1}{2}x^6|_0^1 = \frac{1}{2}$$

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = \frac{1}{2} - \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{1}{50}$$

(d) The variance of X is $\frac{2}{75}$ and the variance of Y is $\frac{1}{25}$. Find the correlation of X and Y.

$$\begin{aligned}\rho(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma_x\sigma_y}\\ &= \frac{1/50}{\sqrt{2/75}\sqrt{1/25}}\\ &= 0.6124\end{aligned}$$

(e) Find the variance of $X+Y$.

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = \frac{2}{75} + \frac{1}{25} + 2\frac{1}{50} = \frac{8}{75}$$

3. Let continuous X have the following PF:

$$f_X(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \sqrt{X}$. Find the PF of Y.

$$F(Y) = P(Y \le y) = P(\sqrt{X} < y) = P(X < y^2) = F_X(y^2).$$

$$\frac{d}{dy}F_X(y^2) = f_X(y^2)(2y) = 4(y^2)^3(2y) = 8y^7.$$

$$f_Y(y) = \begin{cases} 8y^7 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$