Probability Theory: Activity 15 Solutions

1. Let X and Y be discrete random variables with the following joint PF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{7}(x-y)^2 & \text{for } x = 0,1,2 \text{ and } y = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Recall from activity 14 that the marginal PF of X is $f_X(x) = \frac{1}{7}\left(x^2 + (x-1)^2\right)$.

(a) Find the marginal PF of Y

$$f_Y(y) = \sum_x \frac{1}{7}(x-y)^2 = \frac{1}{7}\left(y^2 + (1-y)^2 + (2-y)^2\right)$$

(b) Are X and Y independent?
No, they are not independent, because $f(x,y)$ does not equal $f_X(x)f_Y(y)$.

(c) Find $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{7}(x-y)^2}{\frac{1}{7}(y^2 + (1-y)^2 + (2-y)^2)}$$

$$f_{X|Y}(x|y) = \frac{(x-y)^2}{y^2 + (1-y)^2 + (2-y)^2} \quad \text{for } x = 0,1,2 \text{ and } y = 0,1$$

(d) Find $P(X = 2|Y = 1)$.

$$f_{X|Y}(2|1) = \frac{(2-1)^2}{1 + (1-1)^2 + (2-1)^2} = \frac{1}{2}$$

2. Let X and Y be continuous random variables with the following joint PF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}y^2 & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PF of X:

$$f_X(x) = \int_0^1 \frac{3}{2}y^2 dy = \frac{1}{2}y^3\Big|_0^1 = \frac{1}{2}$$

(b) Find the marginal PF of Y:

$$f_Y(x) = \int_0^2 \frac{3}{2}y^2 dx = \frac{3}{4}y^2x^2\Big|_0^2 = 3y^2$$

(c) Are X and Y independent?
Yes, because $f_X(x)f_Y(y) = f_{X,Y}(x,y)$:

$$\frac{1}{2} \times 3y^2 = \frac{3}{2}y^2$$

3. Let X be a discrete random variable and Y be continuous random variable with the following joint PF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{20}y^x & \text{for } x = 0,1,2 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PF of X:

$$f_X(x) = \int_0^2 \frac{3}{20}y^x dy = \int_0^2 \frac{3}{20(x+1)}y^{x+1}\Big|_0^2 = \frac{3}{20(x+1)}2^{x+1}$$

(b) Find the marginal PF of Y:

$$f_Y(y) = \sum_x \frac{3}{20}y^x = \frac{3}{20}\left(1 + y + y^2\right)$$

(c) Are X and Y independent?
No, because $f_{X,Y}(x,y)$ does not equal $f_X(x)f_Y(y)$.

(d) Find $f_{Y|X}(y|x)$.

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{y^x}{\frac{2^{x+1}}{x+1}}$$

(e) Find $P(0 \le y \le 1|x = 0)$.

$$f_{Y|X=0} = \frac{1}{2}$$

$$P(0 \le Y \le 1|X = 0) = \int_0^1 \frac{1}{2}dy = \frac{1}{2}y\Big|_0^1 = \frac{1}{2}.$$