Probability Theory: Activity 13 Solutions

1. Let X have the following PF:

$$f(x) = \begin{cases} 0.1 & X = 0 \\ 0.3 & X = 1 \\ 0.6 & X = 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $M_x(t)$.

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{2} e^{tx} f(x) = 0.1e^{0t} + 0.3e^{t} + 0.6e^{2t}$$

$$M_x(t) = 0.1 + 0.3e^{t} + 0.6e^{2t}$$

   (b) Using $M_x(t)$, find E(X).

$$M_x'(t) = 0.3e^{t} + 2(.6)e^{2t}$$

$$M_x'(0) = 0.3 + 1.2 = 1.5 = E(X).$$

   (c) Let $Y = 2X + 3$. Use $M_x(t)$ to find $M_y(t)$.

$$M_y(t) = e^{3t}M_x(2t) = 0.1e^{3t} + .3e^{3t}e^{2t} + 0.6e^{3t}e^{2(2t)} = 0.1e^{3t} + .3e^{5t} + 0.6e^{7t}$$

2. Let X follow an exponential distribution with a mean of 1:

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $M_x(t)$.

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}e^{-x}dx = \int_0^\infty e^{-(1-t)x}dx = -\frac{1}{1-t}e^{-(1-t)x}\Big|_0^\infty$$

$$M_x(t) = \frac{1}{1-t}$$

   (b) Let $Y = \frac{X}{\lambda}$, so that Y follows an exponenential with a mean of $\frac{1}{\lambda}$. Use $M_x(t)$ to find $M_y(t)$.

$$M_y(t) = E(e^{t\frac{x}{\lambda}}) = M_x(\frac{t}{\lambda}) = \frac{1}{1-\frac{t}{\lambda}} = \frac{\lambda}{\lambda - t}.$$

   (c) Use $M_y(t)$ to find the mean and variance of of the exponential.

$$M_y(t) = \lambda(\lambda - t)^{-1}$$

$$M_y'(t) = \lambda(\lambda - t)^{-2}$$

$$M_y''(t) = 2\lambda(\lambda - t)^{-3}$$

$$E(X) = M_y'(0) = \lambda(\lambda - 0)^{-2} = \frac{1}{\lambda}.$$

$$E(X^2) = M_y''(0) = 2\lambda(\lambda - t)^{-3} = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - E(X^2) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$