

Probability Theory: Activity 10 Solutions

Consider a probability density function as below:

$$f(x) = c(1 - (1 - x)^2), \text{ for } 0 < x < 2$$

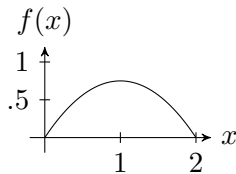
Find the following:

1. Find c and sketch the PF.

We want to set c such that $\int_0^2 c(1 - (1 - x)^2) = 1$, since the integral of a PF must be 1. So we have:

$$\begin{aligned} \int_0^2 c(1 - (1 - x)^2) &= \int_0^2 c(-x^2 + 2x)dx \\ &= c\left(-\frac{x^3}{3} + x^2\right)\Big|_0^2 \\ &= c\left(-\frac{8}{3} + 4\right) \\ &= c\left(\frac{4}{3}\right) \end{aligned}$$

and so c is $\frac{3}{4}$. We can sketch the PF - note that $f(0) = 0$, $f(2) = 0$, and $f(1) = 3/4$. The $-x^2$ tells us this an upside down parabola. So we have:



2. Find $P(1/4 < X < 3/4)$

We just need to integrate from $1/4$ to $3/4$:

$$\begin{aligned} \int_{1/4}^{3/4} \frac{3}{4}(1 - (1 - x)^2)dx &= \int_{1/4}^{3/4} \frac{3}{4}(-x^2 + 2x)dx \\ &= \frac{3}{4}\left(-\frac{x^3}{3} + x^2\right)\Big|_{1/4}^{3/4} \\ &= \frac{3}{4}\left(-\frac{3^2}{4^3} + \left(\frac{3}{4}\right)^2\right) - \frac{3}{4}\left(-\frac{1}{3 \times 4^3} + \left(\frac{1}{4}\right)^2\right) \\ &= 0.2734 \end{aligned}$$

3. Find $P(X < 1/2)$

Once again we need to integrate. Using what we found above, we have:

$$\begin{aligned} = P(X < 1/2) &= \frac{3}{4}\left(-\frac{x^3}{3} + x^2\right)\Big|_0^{1/2} \\ &= \frac{3}{4}\left(-\frac{1}{2^3 \times 3} + \left(\frac{1}{2}\right)^2\right) \\ &= 0.15625 \end{aligned}$$

Following what we did above, we have:

$$\begin{aligned} = P(X < x) &= \frac{3}{4} \left(-\frac{x^3}{3} + x^2 \right) \Big|_0^x \\ &= \frac{3}{4} \left(-\frac{x^3}{3} + x^2 \right) \\ &= \frac{3}{4} x^2 - \frac{x^3}{4} \end{aligned}$$

4. Find $E(X)$.

$$\begin{aligned} E(X) &= \int_0^2 x \frac{3}{4} (-x^2 + 2x) dx \\ &= \int_0^2 \frac{3}{4} (-x^3 + 2x^2) dx \\ &= \frac{3}{4} \left(\frac{-x^4}{4} + \frac{2x^3}{3} \right) \Big|_0^2 \\ &= \frac{3}{4} \left(-4 + \frac{16}{3} \right) = 1. \end{aligned}$$

5. Find $E(X^2)$

$$\begin{aligned} E(X^2) &= \int_0^2 x^2 \frac{3}{4} (-x^2 + 2x) dx \\ &= \int_0^2 \frac{3}{4} (-x^4 + 2x^3) dx \\ &= \frac{3}{4} \left(\frac{-x^5}{5} + \frac{2x^4}{4} \right) \Big|_0^2 \\ &= \frac{3}{4} \left(\frac{-32}{5} + \frac{32}{4} \right) = 1.2 \end{aligned}$$

6. Find $\text{Var}(X)$

$$\text{Var}(X) = E(X^2) - E(X)^2 = 1.2 - (1)^2 = 0.2.$$