Probability Theory: Activity 4 Solutions

Axioms & Properties of Probability

1. Show that for any events A and B, $P(A)+P(B)-1 \leq P(A\cap B) \leq P(A\cup B) \leq P(A)+P(B)$. For each of these three inequalities, give a simple criterion for when the inequality is actually an equality (e.g., give a simple condition such that $P(A\cap B) = P(A\cup B)$ if and only if the condition holds).

   Solution:
   (a) Inequality can be demonstrated using the Inclusion-Exclusion Property,
   $P(A\cup B) = P(A) + P(B) - P(A\cap B)$ and the All-or-Nothing Axiom, $P(S) = 1$.
   $P(A\cup B) = P(A) + P(B) - P(A\cap B) \leq P(S) = 1$ which implies
   $P(A) + P(B) - 1 \leq P(A\cap B)$. Strict equality holds if and only if $A\cup B = S$,
   where S is the sample space.

   (b) Since $A\cap B \subseteq A\cup B$, then $P(A\cap B) \leq P(A\cup B)$ by the Subset Property.
   Strict equality holds if and only if $A = B$.

   (c) Inequality follows directly from the Inclusion-Exclusion Property with strict equality if and only if $P(A\cap B) = 0$.

2. Let A,B be events. Difference $B - A$ is defined to be the set of all elements of B that are not in A. Show that if $A \subseteq B$, then $P(B-A) = P(B) - P(A)$, directly using the axioms of prob.

   Solution: Since $B = (B-A)\cup A$, then $P(B) = P(A) + P(B-A)$ by the Disjoint Axiom of probability. Rearranging terms, $P(B-A) = P(B) - P(A)$.

3. Events A and B are independent if $P(A\cap B) = P(A)P(B)$ (independence is explored in detail in the next chapter). Show that if A and B are independent, then:
   $P(A\cup B) = P(A) + P(B) - P(A)P(B) = 1 - P(A^c)P(B^c)$.

   Solution: $$\begin{aligned}P(A\cup B) &= P(A) + P(B) - P(A\cap B) = P(A) + P(B) - P(A)P(B)\\ &= P(A)(1-P(B)) + P(B) = P(A)P(B^c) + P(B)\\ &= P(A)P(B^c) + 1 - P(B^c) = 1 + P(B^c)(P(A)-1) = 1 - P(B^c)P(A^c).\end{aligned}$$

Inclusion-Exclusion

4. A fair 6-sided die is rolled n times. What's the prob that at least 1 of the 6 sides never appears?

Solution: Let A_i be the event that i is never rolled for $1 \leq i \leq 6$.

The event of interested then is $\bigcup_{i=1}^{6} A_i$. By Inclusion-Exclusion, $P(\bigcup_{i=1}^{6} A_i) = \sum_{i=1}^{6} P(A_i) - \sum_{1\leq i<j\leq 6} P(A_i \cap A_j) + \sum_{1\leq i<j<k\leq 6} P(A_i \cap A_j \cap A_k) - ... - P(\bigcap_{i=1}^{6} A_i)$.

Note that, $P(A_i) = (\frac{5}{6})^n$

$P(A_i \cap A_j) = (\frac{4}{6})^n$

$P(A_i \cap A_j \cap A_k) = (\frac{3}{6})^n$

$P(A_i \cap A_j \cap A_k \cap A_w) = (\frac{2}{6})^n$

$P(A_i \cap A_j \cap A_k \cap A_w \cap A_z) = (\frac{1}{6})^n$

$P(\bigcap_{i=1}^{6} A_i) = 0$

Thus by symmetry, $P(\bigcup_{i=1}^{6} A_i) = 6(\frac{5}{6})^n - \binom{6}{2}(\frac{4}{6})^n + \binom{6}{3}(\frac{3}{6})^n - \binom{6}{4}(\frac{2}{6})^n + \binom{6}{5}(\frac{1}{6})^n$.