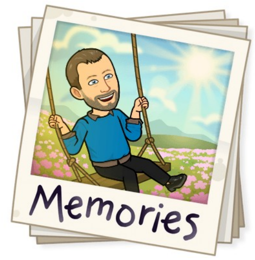
Probability Theory

Textbook: *Introduction to Probability* by Blitzstein and Hwang

Previous Lecture



- Conditional Expectation Given an Event: scalar $E(Y|A)$
- Conditional Expectation Given a rv: rv $E(Y|X)$
- Adam's Law
- Eve's Law

8 - Transformations

§8.1 - Change of Variables

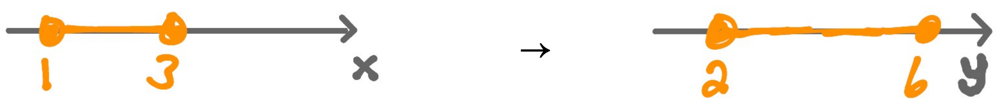
"Change of Variables" is the analog of u-sub in Calc I:

Recall: $\int_1^3 e^{2x}dx =$??

$$= \int_2^6 e^y \tfrac{1}{2}\, dy, \text{ where } \; y = 2x, \quad dy = 2dx, \quad dx = \tfrac{1}{2}dy, \quad 1 \mapsto 2, \quad \text{and} \quad 3 \mapsto 6.$$

$\frac{1}{2}$ is a "scaling factor" from the derivative, or a "fudge factor."

Although we don't emphasize it in Calc I, there is a mapping of intervals here.



Thm (**Change of Vars in 1D**). Let X be cont w/PF f_X, and let $Y := g(X)$, where g is differentiable and strictly increasing (or strictly decreasing). Then the PF of Y is $f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$. The support of Y is all $g(x)$ with x in the support of X.

From our calculus example, we could call X a rv with "PF" $f_X(x) = e^{2x}$,

We let $y := g(x) = 2x$. So $x = g^{-1}(y) = \frac{1}{2}y$, and $\frac{dx}{dy} = \frac{1}{2}$.

Notice that after the transformation we have: $f_Y(y) = e^y \frac{1}{2} = e^{2x}\frac{dx}{dy} = f_X(x)\left|\frac{dx}{dy}\right|$.

Proof. Let X be cont w/PF f_X, and let $Y := g(X)$ be strictly increasing. The CDF of Y is:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

$$= P(X \le g^{-1}(y)) \qquad \text{(the requirement of strictly increasing allow us to take an inverse)}$$

$$= F_X(g^{-1}(y))$$

$= F_X(x);$

so by the chain rule, the PF of Y is $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(x) = f_X(x) \frac{dx}{dy}$.

The proof for g strictly decreasing is analogous. In that case the PF ends up as $-f_X(x) \frac{dx}{dy}$, which is nonnegative since $\frac{dx}{dy} < 0$ if g is strictly decreasing.
Using $\left| \frac{dx}{dy} \right|$, as in the statement of the thm, covers both cases. ■

Easy to remember form (for strictly increasing): $f_Y(y)dy = f_X(x)dx$

Ex (Log-Normal PF): Let $X \sim \mathcal{N}(0, 1)$ and $Y := e^X$. Use change of variables to find the PF of Y.

Note $g(x) = e^x$ is strictly increasing. Let $y = e^x$.

So, $x = \ln y = g^{-1}(y)$ and $\frac{dx}{dy} = \frac{1}{y}$.

Recall $f_X(x) = \varphi(x)$ for $\mathcal{N}(0, 1)$. We want $f_Y(y)$.

Thus, $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \varphi(\ln y) \frac{1}{y}$

Note we specify the support of the distr.
(since x ranges from $-\infty$ to ∞, e^x ranges from 0 to ∞.)

$f_Y(y) = \varphi(\ln y) \frac{1}{y}$, for $y > 0$.

Alternatively, $F_Y(y) = P(Y \leq y)$

$= P(e^X \leq y) = P(X \leq \ln y) = \Phi(\ln y)$.

So, the PF is again $f_Y(y) = \frac{d}{dy} \Phi(\ln y) = \varphi(\ln y) \frac{1}{y}$, for $y > 0$. □

Harvard Video (ignore convolutions): youtube.com/watch?v=yXwPUAIvFyg&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=23

What did we learn?

- Change of Vars in 1D